Representation Theory

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Contents

0 Introduction

Representaiton theory is the theory of how *groups* act as groups of linear transformations on *vector spaces*.

Here the groups are either *finite*, or *compact topological groups* (infinite), for example, $SU(n)$ and $O(n)$. The vector spaces we conside are finite dimensional, and usually over $\mathbb{C}$. Actions are *linear* (see below).

Some books: James-Liebeck (CUP); Alperin-Bell (Springer); Charles Thomas, *Representations of finite and Lie groups*; Onlne notes: SM, Teleman; P.Webb *A course in finite group representation theory* (CUP); Charlie Curtis, *Pioneers of representation theory* (history).

1 Group actions

Throughout this course, if not specified otherwise:
• F is a field, usually $\mathbb{C}$, $\mathbb{R}$ or $\mathbb{Q}$. When the field is one of these, we are discussing *ordinary representation theory*. Sometimes $F = F_p$ or $\bar{F}_p$ (algebraic closure, see Galois Theory), in which case the theory is called *modular representation theory*;
• V is a vector space over F, always finite dimensional;
$GL(V) = \{\theta : V \to V, \theta \text{ linear, invertible}\}$, i.e. $\det\theta \neq 0$.

Recall from Linear Algebra:
If $\dim_F V = n < \infty$, choose basis $e_1, ..., e_n$ over F, so we can identify it with F^n. Then $\theta \in GL(V)$ corresponds to an $n \times n$ matrix $A_\theta = (a_{ij})$, where $\theta(e_j) = \sum_i a_{ij} e_i$. In fact, we have $A_\theta \in GL_n(F)$, the general linear group.

(1.1) $GL(V) \cong GL_n(F)$ as groups by $\theta \to A_\theta$ ($A_{\theta_1\theta_2} = A_{\theta_1}A_{\theta_2}$ and bijection). Choosing different basis gives different isomorphism to $GL_n(F)$, but:

(1.2) Matrices A_1, A_2 represent the same element of $GL(V)$ w.r.t different bases iff they are conjugate (similar), i.e. $\exists X \in GL_n(F)$ s.t. $A_2 = XA_1X^{-1}$.

Recall that $\mathrm{tr}(A) = \sum_i a_{ii}$ where $A = (a_{ij})$, the *trace* of A.

(1.3) $\mathrm{tr}(XAX^{-1}) = \mathrm{tr}(A)$, hence we can define $\mathrm{tr}(\theta) = \mathrm{tr}(A_{\theta_1})$ independent of basis.

(1.4) Let $\alpha \in GL(V)$ where V in f.d. over $\mathbb{C}$, with $\alpha^m = \iota$ for some m (here ι is the identity map). Then α is diagonalisable.

Recall $EndV$ is the set of all ilnear maps $V \to V$, e.g. $End(F^n) = M_n(F)$ some $n \times n$ matrices.

(1.5) *Proposition.* Take V f.d. over $\mathbb{C}$, $\alpha \in End(V)$. Then α is diagonalisable iff there exists a polynomial f with distinct linear factors with $f(\alpha) = 0$. For example, in (1.4), where $\alpha^m = \iota$, we take $f = X^m - 1 = \prod_{j=0}^{m-1}(X - \omega^j)$ where $\omega = e^{2\pi i/m}$ is the (m^{th}) root of unity. In fact we have:

(1.4)* A finite family of commuting separately diagonalisable automorphisms of a $\mathbb{C}$-vector space can be simultaneously diagonalised (useful in abelian groups).

Recall from Group Theory:
(1.6) The symmetric group, $S_n = Sym(X)$ on the set $X = \{1, ..., n\}$ is the set of all permutations of X. $|S_n| = n!$. The alternating group A_n on X is the set of products of an even number of transpositions (2-cycles). $|A_n| = \frac{n!}{2}$.

(1.7) Cyclic groups of order m: $C_m =< x : x^m = 1 >$. For example, $(\mathbb{Z}/m\mathbb{Z}, +)$; also, the group of m^{th} roots of unity in $\mathbb{C}$ (inside $GL_1(\mathbb{C}) = \mathbb{C}^*$, the multiplicative group of $\mathbb{C}$). We also have the group of rotations, centre O of regular $m-$gon in $\mathbb{R}^2$ (inside $GL_2(\mathbb{R})$).

(1.8) Dihedral groups D_{2m} of order $2m =< x, y : x^m = y^2 = 1, yxy^{-1} = x^{-1} >$. Think of this as the set of rotations and reflections preserving a regular m-gon.

(1.9) Quaternion group, $Q_8 =< x, y|x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1} >$ of order 8. For example, in $GL_2(\mathbb{C})$, put $i = \binom{i\ 0}{0\ i}, j = \binom{0\ \ 1}{-1\ 0}, k = \binom{0\ i}{i\ 0}$, then $Q_8 = \{\pm I_2, \pm i, \pm j, \pm k\}$.

(1.10) The conjugacy class (ccls) of $g \in G$ is $\mathcal{C}_G(g) = \{xgx^{-1} : x \in G\}$. Then $|\mathcal{C}_G(g)| = |G : C_G(g)|$, where $C_G(g) = \{x \in G : xg = gx\}$, the centraliser of $g \in G$.

(1.11) Let G be a group, X be a set. G acts on X if there exists a map $\cdot : G \times X \to X$ by $(g, x) \to g \cdot x$ for $g \in G$, $x \in X$, s.t. $1 \cdot x = x$ for all $x \in X$, $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G, x \in X$.

(1.12) Given an action of G on X, we obtain a homomorphism $\theta : G \to Sym(X)$, called the *permutation representation* of G.

Proof. For $g \in G$, the function $\theta_g : X \to X$ by $x \to gx$ is a permutation on X, with inverse $\theta_{g^{-1}}$. Moreover, $\forall g_1, g_2 \in G$, $\theta_{g_1 g_2} = \theta_{g_1}\theta_{g_2}$ since $(g_1g_2)x = g_1(g_2x)$ for $x \in X$. $\square$

2 Basic Definitions

2.1 Representations

Let G be finite, F be a field, usually $\mathbb{C}$.

Definition. (2.1)
Let V be a f.d. vector space over F. A (linear, in some books) *representation* of G on V is a group homomorphism

$$\rho = \rho_V : G \quad \to GL(V)$$

Write ρ_g for the image $\rho_V(g)$; so for each $g \in G$, $\rho_g \in GL(V)$, and $\rho_{g_1 g_2} = \rho_{g_1}\rho_{g_2}$, and $(\rho_g)^{-1} = \rho_{g^{-1}}$.
The *dimension* (or *degree*) of ρ is $\dim_F V$.

(2.2) Recall $\ker \rho \triangleleft G$ (kernel is a normal subgroup), and $G/\ker \rho \cong \rho(G) \leq GL(V)$ (1st isomorphism theorem). We say ρ is *faithful* if $\ker \rho = 1$.

An alternative (and equivalent) approach is to observe that a representation of G on V is "the same as" a *linear action* of G:

Definition. (2.3)
G acts linearly on V if there exists a *linear action*

$$G \times V \to V$$
$$(g, v) \to gv$$

By linear action we mean: (action) $(g_1 g_2)v = g_1(g_2 v)$, $1v = v$ $\forall g_1, g_2 \in G, v \in V$, and (linear) $g(v_1 + v_2) = gv_1 + gv_2$, $g(\lambda v) = \lambda gv$ $\forall g \in G, v_1, v_2 \in V, \lambda \in F$.
Now if G acts linearly on V, the map

$$G \to GL(V)$$
$$g \to \rho_g$$

with $\rho_g : v \to gv$ is a representation of G. Conversely, given a representation $\rho : G \to GL(V)$, we have a linear action of G on V via $g \cdot v := \rho(g)v$ $\forall v \in V, g \in G$.

(2.4) In (2.3) we also say that V is a G-space or that V is a G-module. In fact if we define the *group algebra* FG, or $F[G]$, to be $\{\sum \alpha_j g : \alpha_j \in F\}$ with natural addition and multiplication, then V is actually a FG-module (in the sense from GRM).

(2.5) R is a *matrix representation* of G of degree n if R is a homomorphism $G \to GL_n(F)$. Given representation $\rho : G \to GL(V)$ with $\dim_F V = n$, fix basis B; we get matrix representation

$$G \to GL_n(F)$$
$$g \to [\rho(g)]_B$$

Conversely, given matrix representation $R : G \to GL_n(F)$, we get representation

$$\rho : G \to GL(F^n)$$
$$g \to \rho_g$$

via $\rho_g(v) = R_g v$ where R_g is the matrix of g.

Example. (2.6)
Given any group G, take $V = F$ the 1-dimensional space, and

$$\rho : G \to GL(F)$$
$$g \to (id : F \to F)$$

is known as the trivial representation of G. So $\deg \rho = 1$ ($\dim_F F = 1$).

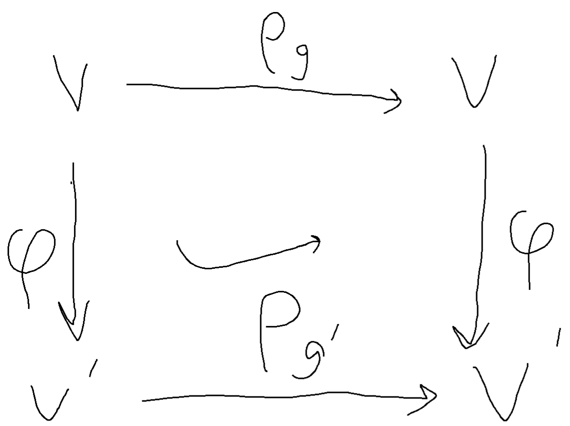
Example. (2.7)
Let $G = C_4 =< x : x^4 = 1 >$. Let $n = 2$, and $F = \mathbb{C}$. Note that any $R : x \to X$ will determine $x^j \to X^j$ as it is a homomorphism, and also we need $X^4 = I$. So we can take X to be diagonal matrix – any such with diagonal entries a root to $x^4 = 1$, i.e. $\{\pm 1, \pm i\}$, or if X is not diagonal then it will be similar to a diagonal matrix by (1.4) ($X^4 = I$).

2.2 Equivalent representations

Definition. (2.8)
Fix G, F. Let V, V' be F-spaces, and $\rho : G \to GL(V)$, $\rho' : G \to GL(V')$ which are representations of G. The linear map $\phi : V \to V'$ is a G-homomorphism if

$$\phi\rho(g) = \rho'(g)\phi \forall g \in G (*)$$

We can understand this more by the following diagram:



We say ϕ *intertwines* ρ, ρ'. Write $Hom_G(V, V')$ for the F-space of all these.
ϕ is a G-isomorphism if it is also bijective; if such ϕ exists, ρ, ρ' are isomorphic/equivalent representations. If ϕ is a G-isomorphism, we can write $(*)$ as $\rho' = \phi\rho\phi^{-1}$.

Lemma. (2.9)
The relation "being isomorphic" is an equivalent relation on the set of all representations of G (over F).

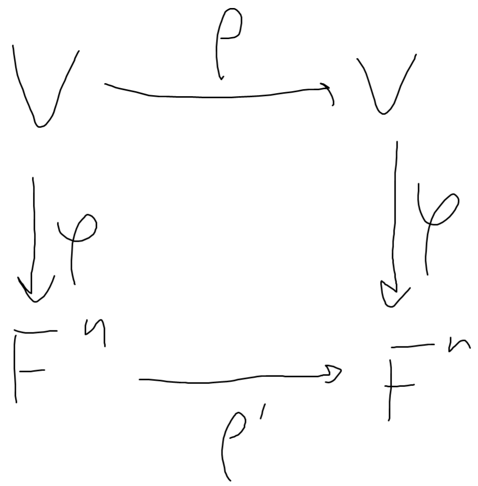
Remark. (2.10)
If ρ, ρ' are isomorphic representations, they have the same dimension.

The converse may be false: C_4 has four non-isomorphic 1-dimensional representations: if $\omega = e^{2\pi i/4}$ then they are $\rho_j(x^i) = \omega^{ij}$ $(0 \leq i \leq 3)$.

Remark. (2.11)
Given G, V over F of dimension n and $\rho : G \to GL(V)$. Fix basis B for V: we get a linear isomorphism

$$\phi : V \to F^n$$
$$v \to [v]_B$$

and we get a representation $\rho' : G \to GL(F^n)$ isomorphic to ρ:



(2.12) In terms of matrix representations, we have

$$R : G \to GL_n(F),$$
$$R' : G \to GL_n(F)$$

are (G)-isomorphic or equivalent if there exists a nonsingular matrix $X \in GL_n(F)$ with $R'(g) = XR(g)X^{-1}$ $\forall g \in G$.

In terms of linear G-actions, the actions of G on V,V' are G-isomorphic if there exists isomorphisms $\phi : V \to V'$ such that $g : \phi(v) = \phi(gv)$ $\forall v \in V, g \in G$.

2.3 Subrepresentations

Definition. (2.13)
Let $\rho : G \to GL(V)$ be a representation of G. We say $W \leq V$ is a G-subspace if it's a subspace and it is $\rho(G)$-invariant, i.e. $\rho_g(W) \leq W \forall g \in G$. Obviously $\{0\}$ and V are G-subspaces, however.
ρ is *irreducible/simple* representation if there are no proper G-subspaces.

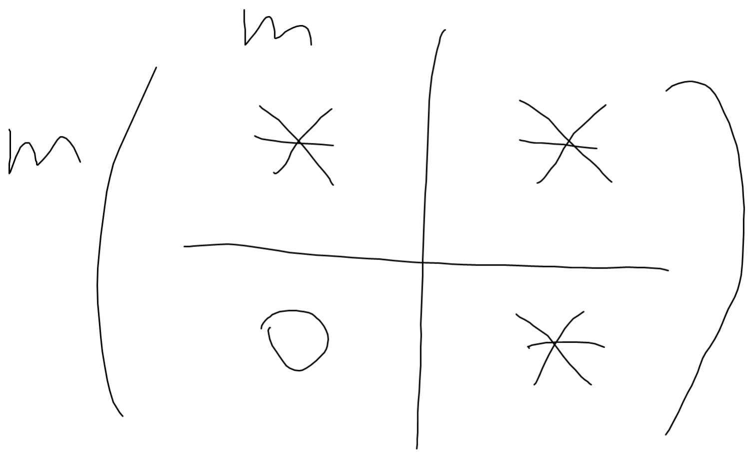
Example. (2.14)
Any 1-dimensional representation of G is irreducible, but not conversely, e.g. D_8 has 2-dimensional $\mathbb{C}$-irreducible representation.

(2.15) In definition (2.13), if W is a G-subspace, then the corresponding map

$$G \to GL(W)$$
$$g \to \rho(g)|_W$$

is a representation of G, a *subrepresentation* of ρ.

Lemma. (2.16)
In definition (2.13), given $\rho : G \to GL(V)$, if W is a G-subspace of V and if $B = \{v_1, ..., v_n\}$ is a basis containing basis $B_1 = \{v_1, ..., v_m\}$ of W $(0 < m < n)$ then the matrix of $\rho(g)$ w.r.t. B has block upper triangular form as the graph below, for



each $g \in G$.

Example. (2.17)
(i) The irreducible representations of $C_4 = \langle x : x^4 = 1 \rangle$ are all 1-dimensional and four of these are $x \to i, x \to -1, x \to -i, x \to 1$. In general, $C_m = \langle x : x^m = 1 \rangle$ has precisely m irreducible complex representations, all of dimension 1. In fact, all complex irreducible representations of a finite abelian group are 1-dimensional (use (1.4)* or see (4.4) below).
(ii) $G = D_6$: any irreducible C-representation has dimension ≤ 2.
Let $\rho : G \to GL(V)$ be irreducible G-representation. Let r, s be rotation and reflection in D_6 respectively. Let V be eigenvector of $\rho(r)$. So $\rho(r)v = \lambda v$

for some $\lambda \neq 0$. Let $W = span\{v, \rho(s)v\} \leq V$. Since $\rho(s)\rho(s)v = v$ and $\rho(r)\rho(s)v = \rho(s)\rho(r)^{-1}v = \lambda^{-1}\rho(s)v$, both of which are in W; so W is G-invariant, i.e. a G-subspace. Since V is irreducible, $W = V$.

Definition. (2.18)
We say at $\rho : G \to GL(V)$ is *decomposable* if there are proper G-invariant subspaces U, W with $V = U \oplus W$. Say ρ is direct sum $\rho_U \oplus \rho_W$. If no such decomposition exists, we say that ρ is *indecomposable*.

Lemma. (2.19)
Suppose $\rho : G \to GL(V)$ is decomposable with G-invariant decomposition $V = U \oplus W$. If B is a basis $\{\underbrace{u_1, ..., u_k}_{B_1}, \underbrace{w_1, ..., w_l}_{B_2}\}$ of V consisting of basis of U and basis of W, then w.r.t. B, $\rho(g)_B$ is a block diagonal matrix $\forall g \in G$ as

$$\rho(g)_B = \begin{pmatrix} [\rho_W(g)]_{B_1} & 0 \\ 0 & [\rho_W(g)]_{B_2} \end{pmatrix}$$

Definition. (2.20)
If $\rho : G \to GL(V)$, $\rho' : G \to GL(V')$, the *direct sum* of ρ, ρ' is

$$\rho \oplus \rho' : G \to GL(V \oplus V')$$

where $\rho \oplus \rho'(g)(v_1 + v_2) = \rho(g)v_1 + \rho'(g)v_2$, a *block diagonal action.* For matrix representations $R : G \to GL_n(F)$, $R' : G \to GL_{n'}(F)$, define $R \oplus R' : G \to GL_{n+n'}(F)$:

$$g \to \begin{pmatrix} R(g) & 0 \\ 0 & R'(g) \end{pmatrix}$$

3 Complete reducibility and Maschke's theorem

Definition. (3.1)
A representation $\rho : G \to GL(V)$ is *completely reducible*, or *semisimple*, if it is a direct sum of irreducible representations. Evidently, irreducible implies completely reducible (lol).

Remark. (3.2)
(1) The converse is false;
(2) See sheet 1 Q3: $\mathbb{C}$-representation of $\mathbb{Z}$ is not completely reducible and also representation of C_p over $\mathbb{F}_p$ is not c.r..

Fron now on, take G finite and *char* $F = 0$.

Theorem. (3.3)
Every f.d. representation V of a finite group over a field of char 0 is completely reducible, i.e.

$$V \cong V_1 \oplus ... \oplus V_r$$

is a direct sum of representations, each V_i irreducible.

It is enough to prove:

Theorem. (3.4 Maschke's theorem, 1899)
Let G be finite, $\rho : G \to GL(V)$ a f.d. representation, *char* $F = 0$. If W is a G-subspace of V, then there exists a G-subspace U of V s.t. $V = W \oplus U$, a direct sum of G-subspaces.

Proof. (1)
Let W' be any *vector subspace* complement of W in V, i.e. $V = W \oplus W'$ as vector spaces, and $W \cap W' = 0$. Let $q : V \to W$ be th projection of V onto W along W' ($\ker q = W'$), i.e. if $v = w + w'$ then $q(v) = w$. Define

$$\bar{q} : v \to \frac{1}{|G|} \sum_{g \in G} gq(g^{-1}v)$$

the 'average' of q over G. Note that in order for $\frac{1}{|G|}$ to exists, we need *char* $F = 0$. It still works if *char* $F \nmid |G|$.
Claim (1): $\bar{q} : V \to W$: For $v \in V$, $g(q^{-1}v) \in W$ and $gW \leq W$;
Claim (2): $\bar{q}(w) = w$ for $w \in W$:

$$\bar{q}(w) = \frac{1}{|G|} \sum_{g \in G} gq(g^{-1}w) = \frac{1}{|G|} \sum g(g^{-1}w) = \frac{1}{|G|} \sum w = w$$

So these two claims imply that $\bar{q}$ projects V onto W.

Claim (3) If $h \in G$ then $h\bar{q}(v) = \bar{q}(hv)$ $(v \in V)$:

$$\begin{aligned} h\bar{q}(v) &= h\frac{1}{|G|}\sum_g g \cdot q(g^{-1}v) \\ &= \frac{1}{|G|}\sum_g hgq(g^{-1}v) \\ &= \frac{1}{|G|}\sum (hg)q((hg)^{-1}hv) \\ &= \frac{1}{|G|}\sum_g gq(g^{-1}(hv)) \\ &= \bar{q}(hv \\ &= \bar{q}(hv)) \end{aligned}$$

We'll then show that the kernel of this map is G-invariant, so this gives a G-summand on Thursday.

Let's now show $\ker \bar{q}$ is G-invariant. If $v \in \ker \bar{q}$, then $h\bar{q}(v) = 0 = \bar{q}(hv)$, so $hv \in \ker \bar{q}$. Thus $V = im\bar{q} \oplus \ker \bar{q} = W \oplus \ker \bar{q}$ is a G-subspace decomposition.

We can deduce (3.3) from (3.4) by induction on $\dim V$. If $\dim V = 0$ or V is irreducible, then result is clear. Otherwise, V has non-trivial G-invariant subspace, W. Then by (3.4), there exists G-invariant complement U s.t. $V = U \oplus W$ as representations of G. But $\dim U, \dim W < \dim V$. So by induction they can be broken up into direct sum of irreducible subrepresentations. □

The second proof uses inner products, hence we need to take $F = \mathbb{C}$ and can be generalised to compact groups in section 15.
Recall, for V a $\mathbb{C}$-space, $\langle , \rangle$ is a *Hermitian inner product* if
(a) $\langle w, v\rangle = \overline{\langle v, w\rangle}$ $\forall v, w$ (Hermitian);
(b) linear in RHS (sesquilinear);
(c) $\langle v, v\rangle > 0$ iff $v \neq 0$ (positvie definite).

Additionally, $\langle , \rangle$ is *G-invariant* if
(d) $\langle gv, gw\rangle = \langle v, w\rangle$ $\forall v, w \in V, g \in G$.

Note if W is G-invariant subspace of V, with G-invariant inner product, then $W^\perp$ is also G-invariant, and $V \oplus W^\perp$. For all $v \in W^\perp$, $g \in G$, we have to show that $gv \in W^\perp$. But $v \in W^\perp \iff \langle v, w\rangle = 0 \forall w \in W$. Thus by (d), $\langle gv, gw\rangle = 0$ $\forall g \in G \forall w \in W$. Hence $\langle gv, w'\rangle = 0$ $\forall w' \in W$. Since we can choose $w = g^{-1}w' \in W$ by G-invariance of W. Thus $gv \in W^\perp$ since g was arbitrary.

Hence if there is a G-invariant inner product on any G-space, we get another proof of Maschke's theorem:

(3.4*) (Weyl's unitary trick)
Let ρ be a complex representation of the finite group G on the $\mathbb{C}$-space V. Then there is a G-invariant Hermitian inner product on V.

Remark. Recall the *unitary group* $U(V)$ on V: $\{f \in GL(V) : (fu, fv) = (u, v) \forall u, v \in V\} = \{A \in GL_n(\mathbb{C}) : A\bar{A}^T = I\} (= U(n))$ by choosing orthonormal

basis.
Sheet 1 Q.12: any finite subgroup of $GL_n(\mathbb{C})$ is conjugate to a subgroup of $U(n)$.

Proof. (2)
There exist an inner product on V: take basis $e_1, ..., e_n$ and define $(e_i, e_j) = \delta_{ij}$, extended sesquilinearly. Now

$$\langle v, w \rangle := \frac{1}{|G|} \sum_{g \in G} (gv, gw)$$

we claim that $\langle , \rangle$ is sesquilinear, positive definite and G-invariant: if $h \in G$, then

$$\langle hv, hw \rangle = \frac{1}{|G|} \sum_{g \in G} ((gh)v, (gh)w)$$
$$= \frac{1}{|G|} \sum_{g' \in G} (g'v, g'w)$$
$$= \langle v, w \rangle$$

for all $v, w \in V$. □

Definition. (3.5, the regular representation)
Recall *group algebra* of G is F-space $FG = span\{e_g : g \in G\}$. There is a linear G-action

$$h \in G, h \sum_{g \in G} a_g e_g = \sum_{g \in G} a_g e_{hg} (= \sum_{g' \in G} a_{h^{-1}g'} e_{g'})$$

ρ_{reg} is the corresponding representation, the *regular representation* of G. This is faithful of dim $|G|$. FG is the *regular module.*

Proposition. Let ρ be an irreducible representation of G over a field of characteristic 0. Then ρ is isomorphic to a subrepresentation of ρ_{reg}.

Proof. Take $\rho : G \in GL(V)$ irreducible and let $0 \neq v \in V$. Let $\theta : FG \to V$ by $\sum a_g e_g \to \sum a_g gv$. Check this is a G-homomorphism. Now V is irreducible so $im\theta = V$ (since $im\theta$ is a G-subspace).

Also $\ker\theta$ is G-subspace of FG. Let W be G-complement of $\ker\theta$ in FG (Maschke), so that $W < FG$ is G-subspace and $FG = \ker\theta \oplus W$. Thus $W \cong FG/\ker\theta \cong (G - isomorphism) im\theta \cong V$. □

More generally,

Definition. (3.7)
Let F be a field. Let G act on set X. Let $FX = span\{e_x : x \in X\}$ with G-action

$$g(\sum a_x e_x) = \sum a_x e_{gx}$$

The representation $G \to GL(V)$ where $V = FX$ is the corresponding *permutation representation.* See section 7.

4 Schur's lemma

It's really unfair that such an important result is only remembered by a lemma, so we shall call it a theorem.

Theorem. (4.1, Schur)
(a) Assume V, W are irreducible G-spaces over field F. Then any G-homomorphism $\theta : V \to W$ is either 0 or an isomorphism.
(b) Assume F is *algebraically closed*, and let V be an irreducible G-space. Then any G-endomorphism $V \to V$ is a scalar multiple of the identity map ι_V.

Proof. (a) Let $\theta : V \to W$ be a G-homomorphism. Then ker θ is G subspace of V and, since V is irreducible, we get $\ker\theta = 0$ or $\ker\theta = V$.
And $im\theta$ is G-subspace of W, so as W is irreducible, $im\theta$ is either 0 or W. Hence, either $\theta = 0$ or θ is injective and surjective, hence isomorphism.
(b) Since F is algebraically closed, θ has an eigenvalue, λ. Then $\theta - \lambda\iota$ is singular G-endomorphism of V, but it cannot be an isomorphism, so it is 0 (by (a)). So $\theta = \lambda\iota_V$. □

Recall from (2.8), the F-space $Hom_G(V, W)$ of all G-homomorphisms $V \to W$. Write $End_G(V)$ for the G-endomorphisms of V.

Corollary. (4.2)
If V, W are irreducible complex G-spaces, then

$$\dim_{\mathbb{C}} Hom_G(V, W) = \begin{cases} 1 & \text{if } V, W \text{ are } G- \text{ isomorphic} \\ 0 & \text{otherwise} \end{cases}$$

Proof. If V, W are not G-isomorphic then the only G-homomorphism $V \to W$ is 0 by (4.1). Assume $v \cong_G W$ and $\theta_1, \theta - 2 \in Hom_G(V, W)$, both non-zero. Then θ_2 is invertible by (4.1), and $\theta_2^{-1}\theta_1 \in End_G(V)$, and non-zero, so $\theta_2^{-1}\theta_1 = \lambda\iota_V$ for some $\lambda \in \mathbb{C}$. Hence $\theta_1 = \lambda\theta_2$. □

Corollary. (4.3)
If finite group G has a faithful complex irreducible representation, then $Z(G)$, the centre of the group, is cyclic.
Note that the converse is false (Sheet 1, Q10).

Proof. Let $\rho : G \to GL(V)$ be faithful irreducible complex representation. Let $z \in Z(G)$, so $zg = gz\ \forall g \in G$, hence the map $\phi_z : v \to z(v)$ $(v \in V)$ is G-endomorphism of V, hence is multiplication by scalar μ_z, say.
By Schur's lemma, $z(v) = \mu_z v\ \forall v$. Then the map

$$\begin{aligned} Z(G) &\to \mathbb{C}^* \text{ (multiplicative group)} \\ z &\to \mu_z \end{aligned}$$

is a representation of Z and is faithful, since ρ is. Thus $Z(G)$ is isomorphic to some finite subgroup of $\mathbb{C}^*$, so is cyclic. □

Let's now consider representation of finite abelian groups.

Corollary. (4.4)
The irreducible $\mathbb{C}$-representations of a finite abelian group are all 1-dimensional.

Proof. Either: use (1.4)* to invoke simultaneous diagonalisation: if v is an eigenvector for each $g \in G$, and if V is irreducible, then $V = \langle v \rangle$.
Or: Let V be an irreducible $\mathbb{C}$-representation. For $g \in G$, the map

$$\begin{aligned} \theta_g : V &\to v \\ v &\to gv \end{aligned}$$

is a G-endomorphism of V, and as V irreducible, $\theta_g = \lambda_g \iota_V$ for some $\lambda_g \in \mathbb{C}$. Thus $gv = \lambda_g v$ for any $g \in G$ (so $\langle v \rangle$ is a G-subspace of V). Thus as $0 \neq V$ is irreducible, $V = \langle v \rangle$, which is 1-dimensional. □

Remark. Schur's lemma fails over non-algebraically closed field, in particular, over $\mathbb{R}$. For example, let's consider the cyclic group C_3. It has 2 irreducible $\mathbb{R}$-representations, one of dimension 1 (maps everything to 1) and one of dimension 2(imo consider $\mathbb{C}$ as a dimension 2 space over $\mathbb{R}$, then map the generator to the 3rd root of unity?) (so 'contradicting' with Schur's lemma via the corollary above).
Recall that every finite abelian group G is isomorphic to a product of cyclic groups (see GRM). For example, $C_6 = C_2 \times C_3$. In fact, it can be written as a product of C_{p^α} for various primes p and $\alpha \geq 1$, and the factors are uniquely determined up to reordering.

Proposition. (4.5)
The finite abelian group $G = C_{n_1} \times ... \times C_{n_r}$ has precisely $|G|$ irreducible $\mathbb{C}$-representations, as described below:

Proof. Write $G = \langle x_1 \rangle \times ... \langle x_r \rangle$ where $|x_j| = n_j$. Suppose ρ is irreducible, so by (4.4), it's 1-dimensional: $\rho : G \to \mathbb{C}^*$.
Let $\rho(1, ..., x_j, ..., 1)$ (all 1 apart from the j^{th} entry) be λ_j. Then $\lambda_j^{n_j} = 1$, so λ_j is a n_j-th root of unity. Now, the values $(\lambda_1, ..., \lambda_r)$ determine ρ:

$$\rho(x_1^{j_1}, ..., x_r^{j_r}) = \lambda_1^{j_1} ... \lambda_r^{j_r}$$

thus $\rho \leftrightarrow (\lambda_1, ..., \lambda_r)$ with $\lambda_j^{n_j} = 1\ \forall j$; we have $n_1 ... n_r$ such r-tuples, each giving 1-dimensional representation. □

Example. (4.6)
Consider $G = C_4 = \langle x \rangle$. We could have $\rho_1(x) = 1, \rho_2(x) = i, \rho_3(x) = -1, \rho_4(x) = -i$.

Warning: There is no "natural" 1-1 correspondence between the elements of G and the representations of G (G-finite abelian). If you choose an isomorphism $G \cong C_{a_1} \times ... \times C_{a_r}$, then we can identify the two sets (elements of groups and representations of G), but it depends on the choice of isomorphism.

Isotypical decomposition:

Recall any diagonalisable endomorphism $\alpha : V \to V$ gives eigenspace decomposition of $V \cong \oplus_\lambda V(\lambda)$, where $V(\lambda) = \{v : \alpha v = \lambda v\}$. This is *caconical* (one of the three useless words: *arbitrary*(anything), *canonical*(only one choice), *uniform*(you can choose, but it doesn't really matter)), in the sense that it depends on α alone (and nothing else).
There is no canonical eigenbasis of V: must choose basis in each $V(\lambda)$.

We know that in *char* 0 every representation V decomposes as $\oplus n_i V_i$, V_i irreducible, $n_i \geq 0$. How unique is this?

We have this wishlist (4.7):
(a) Uniqueness: for each V there is only one way to decompose V as above. However, this doesn't work obviously.
(b) Isotypes: for each V, there exists a unique collection of subrepresentations $U_1, ..., U_k$ s.t. $V = \oplus U_i$ and, if $V_i \subseteq U_i$ and $V_j' \subseteq U_j$ are irreducible subrepresentations, then $V_i \cong V_j'$ iff $i = j$.
(c) Uniqueness of factors: If $\oplus_{i=1}^k V_i \cong \oplus_{i=1}^k V_i'$ with V_i, V_i' irreducible, then $k = k'$, and $\exists \pi \in S_k$ such that $V_{\pi(i)}' \cong V_i$ (Krull-Schimdt theorem). For (b),(c) see Teleman section 5.

Lemma. (4.8)
Let V, V_1, V_2 be $G-$spaces over F.
(i) $Hom_G(V, V_1 \oplus V_2) \cong Hom_G(V, V_1) \oplus Hom_G(V, V_2)$;
(ii) $Hom_G(V_1 \oplus V_2, V) \cong Hom_G(V_1, V) \oplus Hom_G(V_2, V)$;

Proof. (i) Let $\pi_i : V_1 \oplus V_2 \to V_i$ be G-linear projections onto V_i, with kernel V_{3-i} $(i = 1, 2)$.
Consider

$$Hom_G(V, V_1 \oplus V_2) \to Hom_G(V, V_1) \oplus Hom_G(V, V_2)$$
$$\phi \to (\pi_1\phi, \pi_2\phi)$$

This map has inverse $(\psi_1, \psi_2) \to \psi_1 + \psi_2)$. Check details.
(ii) The map $\phi \to (\phi|_{V_1}, \phi|_{V_2})$ has inverse $(\psi_1, \psi_2) \to \psi_1\pi_1 + \psi_2\pi_2$. □

Lemma. Let F be algebraically closed, $V = \oplus_1^n V_i$ a decomposition of G-space into irreducible summands. Then, for each irreducible representation S of G,

$$\#\{j : V_j \cong S\} = \dim Hom_G(S, V)$$

where # means 'number of times'. This is called the *multiplicity* of S in V.

Proof. Indunction on n. $n = 0, 1$ are trivial.
If $n > 1$, $V = \oplus_1^{n-1} V_i \oplus V_n$. By (4.8) we have

$$\dim Hom_G(S, \oplus_1^{n-1} V_i \oplus V_n) = \dim Hom(S, \oplus_1^{n-1} V_i) + \underbrace{\dim Hom_G(S, V_n)}_{\text{Schur's lemma}}$$

□

Definition. (4.10)
A decomposition of V as $\oplus W_j$ where each $W_j \cong n_j$ copies of irreducible representations S_j (each non-isomorphic for each j) is the *canonical decomposition* or the decomposition into *isotypical components* W_j. For F algebraically closed, $n_j = \dim Hom_G(S_j, V)$.

5 Character theory

We want to attach invariants to representation ρ of a finite group G on V. Matrix coefficients of $\rho(g)$ are basis dependent, so not true invariants.
Let's take $F = \mathbb{C}$, G finite, $\rho = \rho_V : G \to GL(V)$ be a representation of G.

Definition. (5.1)
The *character* $\chi_\rho = \chi_V = \chi$ is defined as $\chi(g) = \operatorname{tr}\rho(g) = \operatorname{tr} R(g)$ where $R(g)$ is any matrix representation of $\rho(g)$ w.r.t. any basis.
The degree of χ_V is $\dim_{\mathbb{C}} V$.
Thus χ is a function $G \to \mathbb{C}$. χ is *linear* (not a universal name) if $\dim V = 1$, in which case χ is a homomorphism $G \to \mathbb{C}^*$ $(= GL_1(\mathbb{C}))$.
χ is irreducible if ρ is; χ is faithful if ρ is; and, χ is trivial, or principal, if ρ is the trivial representation (2.6). We write $\chi = 1_G$ in that case.
χ is a complete invariant in the sense that it determines ρ up to isomorphism – see (5.7).

Theorem. (5.2, first properties)
(i) $\chi_V(1) = \dim_{\mathbb{C}} V$; (clear: $\operatorname{tr} I_n = n$)
(ii) χ_V is a *class function*, via it is conjugation-invariant:

$$\chi_V(hgh^{-1}) = \chi_V(g) \forall g, h \in G$$

Thus χ_V is constant on conjugacy classes.
(iii) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$, the complex conjugate;
(iv) For two representations V, W, $\chi_{V\oplus W} = \chi_V + \chi_W$.

Proof. (ii) $\chi(hgh^{-1}) = \operatorname{tr}(R_h R_g R_h^{-1}) = \operatorname{tr}(R_g) = \chi(g)$.
(iii) Recall $g \in G$ has finite order, so we can assume $\rho(g)$ is represented by a diagonal matrix $Diag(\lambda_1, ..., \lambda_n)$. Then $\chi(g) = \sum \lambda_i$. Now g^{-1} is represented by the matrix $Diag(\lambda_1^{-1}, ...\lambda_n^{-1})$, and hence $\chi(g^{-1}) = \sum \lambda_i^{-1} = \sum \bar{\lambda}_i = \overline{\chi(g)}$ (since λ_i's are roots of unity – since $g^k = 1$ for some k!(I mean an exclamation mark here to express surprise) and by homomorphism we know that).
(iv) Suppose $V = V_1 \oplus V_2$, $\rho_i : G \to GL(V_i)$, $\rho : G \to GL(V)$. Take basis $B = B_1 \cup B_2$ of V w.r.t B, $\rho(g)$ has matrix of block form $Diag([\rho_1(g)]_{B_1}, [\rho_2(g)]_{B_2})$ and as $\chi(g)$ is the trace of the above matrix, it is equal ot $\operatorname{tr}\rho_1(g) + \operatorname{tr}\rho_2(g) = \chi_{\rho_1}(g) + \chi_{\rho_2}(g)$. □

Remark. We see later that χ_1, χ_2 character of G implies that $\chi_1\chi_2$ is also a character of G: uses tensor products, see (9.6).

Lemma. (5.3)
Let $\rho : G \to GL(V)$ be a copmlex representation *affording* the character χ (i.e. χ is a character of ρ). Then $|\chi(g)| \le \chi(1)$, with equality iff $\rho(g) = \lambda_I$ for some $\lambda \in \mathbb{C}$, a root of unity. Moreover, $\chi(g) = \chi(1)$ iff $g \in \ker \rho$.

Proof. Fix g. W.r.t. basis of V of eigenvalues $\rho(g)$, the matrix of $\rho(g)$ is $Diag(\lambda_1, ..., \lambda_n)$. Hence $|\chi(g)| = |\sum \lambda_j| \le \sum |\lambda_j| = \sum 1 = \dim V = \chi(1)$. Equality holds iff all λ_j are equal (to λ, say).
If $\chi(g) = \chi(1)$, then $\rho(g) = \lambda\iota$ has $\chi(g) = \lambda\chi(1)$. □

Lemma. (5.4)
(a) If χ is a complex irreducible character of G, so is $\bar{\chi}$;
(b) Under the same assumption, so is $\varepsilon\chi$ for any linear character ε of G.

Proof. If $R : G \to GL_n(\mathbb{C})$ is a complex irreducible representation then so is $\bar{R} : G \to GL_n(\mathbb{C})$ by $g \to \bar{R}(g)$. Similarly for $R' : g \to \varepsilon(g)R(g)$ for $g \in G$. Check the details. □

Definition. (5.5)
$\mathcal{C}(G) = \{f : G \to \mathbb{C} : f(hgh^{-1}) = f(g) \forall h, g \in G\}$, the $\mathbb{C}$-space of class functions (we call it a space since $f_1 + f_2 : g \to f_1(g) + f_2(g)$, $\lambda f : g \to \lambda f(g)$ are still in $\mathcal{C}(G)$), so this is a vector space.
Let $k = k(G)$ be the number of ccls of G. List the ccls $\mathcal{C}_1, ..., \mathcal{C}_k$. Conventionally we choose $g_1 = 1, g_2, ..., g_k$, representatives of the ccls (hence $\mathcal{C}_1 = \{1\}$). Note that $\dim_{\mathbb{C}} \mathcal{C}(G) = k$ (the characteristic functions δ_j of each ccl which maps any element in the ccl to 1 and others to 0 form a basis).
We define Hermitian inner product on $\mathcal{C}(G)$:

$$\begin{aligned}\langle f, f' \rangle &= \frac{1}{|G|} \sum_{g \in G} \overline{f(G)} f'(g) \\ &= \frac{1}{|G|} \sum_{j=1}^{k} |\mathcal{C}_j| \overline{f(g_j)} f'(g_j) \\ &= \sum_{j=1}^{k} \frac{1}{|C_G(g_j)} \overline{f(g_j)} f'(g_j)\end{aligned}$$

using $|\mathcal{C}_x| = |G : C_g(x)|$, where $\mathcal{C}_x$ is the ccl of x, $C_G(x)$ is the centraliser of x.
For characters

$$\langle \chi, \chi' \rangle = \sum \frac{1}{|C_G(g_j)|} \chi(g_j^{-1}) \chi'(g_j)$$

is a real symmetric form (in fact, $\langle \chi, \chi' \rangle \in \mathbb{Z}$ – see later).

Theorem. (5.6)
The $\mathbb{C}$-irreducible characters of G form an orthonormal basis of $\mathcal{C}(G)$. Moreover,
(a) If $\rho : G \to GL(V), \rho' : G \to GL(V')$ are irreducible representations of G affording characters χ, χ' respecitvely, then

$$\langle \chi, \chi' \rangle = \begin{cases} 1 & \rho, \rho' \text{ are isomorphic representations} \\ 0 & \text{otherwise} \end{cases}$$

we call this 'row orthogonality'.
(b) Each class function of G can be expressed as a linear combination of G.
This will be proved later in section 6.

Corollary. (5.7)
Complex representations of *finite* groups are characterised by their characters.
We emphasise on finiteness here: for example, $G = \mathbb{Z}$, consider $1 \to I_2$, $1 \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are non-isomorphic but have same character.

Proof. Let $\rho : G \to GL(V)$ be representation affording χ (G finite over $\mathbb{C}$). (3.3) says

$$\rho = m_1\rho_1 \oplus ... \oplus m_k\rho_k$$

where $\rho_1, ..., \rho_k$ are irreducible, and $m_j \geq 0$. Then $m_j = \langle \chi, \chi_j \rangle$ where χ_j is afforded by ρ_j: we have $\chi = m_1\chi_1 + ... + m_k\chi_k$, but the ρ_i's are orthonormal. □

Corollary. (5.8, irreduciblility criterion)
If ρ is $\mathbb{C}$-representation of G affording χ, then ρ irreducible $\iff$ $\langle \chi, \chi \rangle = 1$.

Proof. Forward is just the statement of orthonormality. Conversely, assume $bra\chi, \chi\rangle = 1$. Now take a (complete) decomposition of ρ and take characters of it we get $\chi = \sum m_j\chi_j$ with χ_j irreducible and $m_j \geq 0$. Then $\sum m_j^2 = 1$. Hence $\chi = \chi_j$ for some j (since the m_j's are obviously integers), so is irreducible. □

Corollary. (5.9)
If the irreducible $\mathbb{C}$-representations of G are $\rho_1, ..., \rho_k$ have dimensions $n_1, ..., n_k$, then

$$|G| = \sum_{i=1}^{k} n_i^2$$

Proof. Recall from (3.5), $\rho_{reg}; G \to GL(\mathbb{C}G)$, the regular representation G of dimension $|G|$ (where $\mathbb{C}G$ is just a G-space with basis $\{e_g : g \in G\}$ and any $h \in G$ permutes the e_g: $e_g \to e_{hg}$).
Let π_{reg} be its charcter, the *regular character* of G.
Claim 1: $\pi_{reg}(1) = |G|$, $\pi_{reg}(h) = 0$ if $h \neq 1$.
This is clear: take $h \in G, h \neq 1$, then we always have 0 down the diagonal since h permutes things around, so the trace is 0; if $h = 1$ then we have an identity matrix so trace is $\dim\rho = |G|$.
Claim 2: $\pi_{reg} = \sum n_j\chi_j$ with $n_j = \chi_j(1)$.
This is because

$$\begin{aligned} n_j &= \langle \pi_{reg}, \chi_j \rangle \\ &= \frac{1}{|G|}\sum_{g\in G} \overline{\pi_{reg}(g)}\chi_j(g) \\ &= \frac{1}{|G|} \cdot |G|\chi_j(1) = \chi_j(1) \end{aligned}$$

(all the other $\pi_{reg}(g)$ are zero by claim 1).
Our corollary is then obvious by just calculating $|G| = \pi_{reg}(1)$. □

Corollary. (5.10)
Number of irreducible characters of G (up to equivalence) = k (=number of ccls).

Corollary. (5.11)
Elements $g_1, g_2 \in G$ are conjugate iff $\chi(g_1) = \chi(g_2)$ for all irreducible characters of G.

Proof. Forward: characters are class functions;
Backward: Let δ be the characteristic function of the class of g_1. In particular, δ is a class function, so can be written as a linear combination of the irreducible characters of G. Hence $\delta(g_2) = \delta(g_1) = 1$, so $g_2 \in \mathcal{C}_G(g_1)$. □

In the end let's introduce a good friend which will be around for the next few lectures:
Recall from (5.5), the inner product on $\mathcal{C}(G)$ and the real symmetric form $\langle , \rangle$ on characters:

Definition. The *character table* of G is the $k \times k$ matrix (where k is the number of ccls) $X = [\chi_i(g_j)]$, the i^{th} character on the j^{th} class, where we let $\chi_1 = 1_G, \chi_2, ..., \chi_k$ are the irreducible characters of G, and $\mathcal{C}_1 = \{1\}, ..., \mathcal{C}_k$ are the ccls with $g_j \in \mathcal{C}_j$ (as we defined in 5.5).
So the $(i,j)^{th}$ entry of X is just $\chi_i(g_j)$.

Example. (5.13)
(a) $C_3 = \langle x : x^3 = 1\rangle$. The character table is

	1	x	x^2
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

where $\omega = e^{2\pi i/3}$.
(b) $G = D_6 \cong S_3 = \langle r, s : r^3 = s^2 = 1, sr^{-1} = r^{-1}\rangle$.
ccls of G: $\mathcal{C}_1 = \{1\}$, $\mathcal{C}_2 = \{r, r^{-1}$, $\mathcal{C}_3 = \{s, sr, sr^2\}$. We have 3 irreducible representations over $\mathbb{C}$: 1_G (trivial); $\mathcal{S}$ (sign): $x \to 1$ for x even, $x \to -1$ for x odd; and W (2-dimensional): sr^i acts by matrix with eigenvalues ± 1; r^k acts by the matrix

$$\begin{matrix} \cos 2k\pi/3 & -\sin 2k\pi/3 \\ \sin 2k\pi/3 & \cos 2k\pi/3 \end{matrix}$$

so $\chi_w(sr^i) = 0 \ \forall j$, $\chi_w(r^k) = 2\cos 2k\pi/3 = -1 \ \forall k$. So the charactable is:

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
1_G	1	1	1
χ_s	1	-1	1
χ_w	2	0	-1

6 Proofs and orthogonality

We want to prove(5.6): irreducible characters form orthonormal basis for the space of $\mathbb{C}$-class functions.

Proof. (of 5.6 (a))
Fix bases of V and V'. Write $R(g)$, $R'(g)$ for matrices of $\rho(g), \rho'(g)$ w.r.t. these bases, respectively. Then

$$\begin{aligned}\langle \chi', \chi \rangle &= \frac{1}{|G|}\chi'(g^{-1})\chi(g) \\ &= \frac{1}{|G|}\sum_{g\in G, i,j \ s.t. 1\leq i\leq n', 1\leq j\leq n} R'(g^{-1})_{ii}R(g)_{jj}\end{aligned}$$

the trick is to define something that annhilates almost the whole thing. Let $\phi : V \to V'$ be linear and define

$$\begin{aligned}\tilde{\phi} : V \to & \quad V' \\ v \to & \quad \frac{1}{|G|}\sum_{g\in G}\rho'(g^{-1})\phi\rho(g)v\end{aligned}$$

We claim that this is a G-homomorphism: if $h \in G$, let's calculate

$$\begin{aligned}\rho'(h^{-1})\tilde{\phi}\rho(h)(v) &= \frac{1}{|G|}\sum_{g\in G}\rho'(gh)^{-1}\phi\rho(gh)(v) \\ &= \frac{1}{|G|}\sum_{g'\in G}\rho'(g'^{-1})\phi\rho(g')(v) \\ &= \tilde{\phi}(v)\end{aligned}$$

(when g runs through G, gh runs through G as well). So (2.8) is satisfied, i.e. ϕ is a G-homomorphism.

Case 1: ρ, ρ' are not isomorphic. Schur's lemma says $\tilde{\phi} = 0$ for any given linear $\phi : V \to V'$. Take $\phi - \varepsilon_{\alpha\beta}$, having matrix $E_{\alpha\beta}$ (w.r.t our basis). This is 0 everywhere except 1 in the (α, β)-position. Then $\tilde{\varepsilon_{\alpha\beta}} = 0$. So $\frac{1}{|G|}\sum_{g\in G}(R'(g^{-1})E_{\alpha\beta}R(g))_{ij} = 0$. So $\frac{1}{|G|}\sum R'(G^{-1})_{i\alpha}R(g)_{\beta j} = 0$ $\forall i, j$, with $\alpha = i, \beta = j$. Now $\frac{1}{|G|}\sum_{g\in G} R'(g^{-1})_{ii}R(g)_{jj} = 0$ sum over i, j. Then $\langle \chi', \chi\rangle = 0$.
Case 2: ρ, ρ' isomorphic. So $\chi = \chi'$; take $V = V'$, $\rho = \rho'$. If $\phi : V \to V$ is linear endomorphism, we claim $\operatorname{tr}\phi = \operatorname{tr}\tilde{\phi}$:

$$\operatorname{tr}\tilde{\phi} = \frac{1}{|G|}\sum_{g\in G}\operatorname{tr}(\rho(g)^{-1}\phi\rho(g)) = \frac{1}{|G|}\sum_{g\in G}\operatorname{tr}\phi = \operatorname{tr}\phi$$

By Schur's lemma, $\tilde{\phi} = \lambda\iota_V$ for some $\lambda \in \mathbb{C}$ (depending on ϕ). Then $\lambda = \frac{1}{n}\operatorname{tr}\phi$. Let $\phi = \varepsilon_{\alpha\beta}$. So $\operatorname{tr}\phi = \delta_{\alpha\beta}$. Hence $\tilde{\varepsilon_{\alpha\beta}} = \frac{1}{n}\delta_{\alpha\beta}\iota_v = \frac{1}{|G|}\sum_{g\in G}\rho(g^{-1})\varepsilon_{\alpha\beta}\rho(g)$. In terms of matrices, take (i, j)-entry: $\frac{1}{|G|}\sum_j R(g^{-1})_{i\alpha}R(g)_{\beta j} = \frac{1}{n}\delta_{\alpha\beta}\delta_{ij}$ $\forall i, j$. Put $\alpha = i, \beta = j$ to get $\frac{1}{|G|}\sum_g R(g^{-1})_{ii}R(g)_{jj} = \frac{1}{n}\delta_{ij}$. Finally sum over i, j to get $\langle \chi, \chi\rangle = 1$. □

Before proving (b), let's prove column orthogonality:

Theorem. (6.1, column orthogonality relations)

$$\sum_{i=1}^{k} \overline{\chi_i(g_j)}\chi_i(g_l) = \delta_{jl}|C_G(g_j)|$$

having an easy corollary

Corollary. (6.2)
$|G| = \sum_{i=1}^{k} \chi_i^2(1)$.

Proof. (of (6.1))
$\delta_{ij} = \langle \chi_i, \chi_j \rangle = \sum \overline{\chi_i(g_l)}\chi_j(g_l)/|C_G(g_l)|$. Consider the character table $X = (\chi_i(g_j))$. Then $\bar{X}D^{-1}X^T = I_{k\times k}$ where $D = Diag(|C_G(g_1)|, ..., |C_G(g_k)|)$.
Since X is quare, it follows that $d6{-}1\bar{X}^T$ is the inverse of X, so $\bar{X}^T X = D$. □

Proof. (of (5.6(b)))
The χ_i generate $\mathcal{C}_G$. Let all the irreducible characters $\chi_1, ..., \chi_l$ of G: claim these generate $\mathcal{C}_G$, the $\mathbb{C}$-space of class functions on G. It's enough to show that the orthogonal complement to $span\{\chi_1, ..., \chi_l\}$ in $\mathcal{C}_G$ is $\{0\}$. To see this, assume $f \in \mathcal{C}_G$ with $\langle f, \chi_j \rangle = 0 \forall j$. Let $\rho : G \to GL(V)$ be irreducible representation affording $\chi \in \{\chi_1, ..., \chi_l\}$. Then $\langle f, \chi \rangle = 0$.
Consider

$$\frac{1}{|G|}\sum_G \overline{f(g)}\rho(g) : V \to V$$

This is a G-homomorphism, so as ρ is irreducible, it must be λ_ι for some $\lambda \in \mathbb{C}$.
Now

$$n\lambda = \text{tr}\,\frac{1}{|G|}\sum_g \overline{f(g)}\rho(g)$$
$$= \frac{1}{|G|}\sum_g \overline{f(g)}\chi(g) = 0 = \langle f, \chi \rangle$$

So $\lambda = 0$. Hence $\sum \overline{f(g)}\rho(g) = 0$, the zero endomorphism on V for all representations ρ (complete reducibility).

Take $\rho = \rho_{reg}$ where $\rho_{reg}(g) : e_1 \to e_g$ $(g \in G)$. So

$$\sum_g \overline{f(g)}\rho_{reg}(g) : e_1 \to \sum_g \overline{f(g)}e_g$$

So it follows $\sum_g \overline{f(g)}e_g = 0$. So $\overline{f(g)} = 0 \forall g \in G$, so $f \equiv 0$. □

Variuous corollaries now follow:
- The number of irreducible representations of G = number of ccls; (5.10)
- Column orthogonality (6.1);
- $|G| = \sum n_i^2$ (6.2);
- $g_1 \tilde{G} g_2 \iff \chi(g_1) = \chi(g_2)$ for all irreducible χ (5.11);
- If $g \in G$, $g \tilde{G} g^{-1} \iff \chi(g) \in \mathbb{R}$ for all irreducible χ.

7 Permutation representations

Preview was given in (3.7). Recall: • G finite group acting on finite set $X = \{x_1, ..., x_n\}$;
• $\mathbb{C}X = \mathbb{C}$-space, with basis $\{e_{x_1}, ..., e_{x_n}\}$ of dimension $|X|$, so is $\{\sum_j a_j e_{x_j} : a_j \in \mathbb{C}\}$;
• corresponding permutation representation $\rho_X : G \to GL(\mathbb{C}X)$ by $g \to \rho(g)$, where $\rho(g)$ sends $e_{x_j} \to e_{gx_j}$, extending linearly.
• ρ_X is the *permutation representation* corresponding to the action of G on X.
• matrices representing $\rho_X(g)$ w.r.t. basis $\{e_x\}_{x\in X}$ are permutation matrices: 0 except for one 1 in each row and column, and $(\rho(g))_{ij} = 1$ iff $gx_j = x_i$. Consider its character:

(7.1) Permutation character, π_X, is

$$\pi_X(g) = |Fix_X(g)| = |\{x \in X : gx = x\}|.$$

(7.2) ρ_X always contains 1_G: $span\{e_{x_1} + ... + e_{x_n}\}$ is a trivial G-subspace of $\mathbb{C}X$ with G-invariant complement $span\{\sum a_x e_x : \sum a_x = 0\}$.

Lemma. (7.3, Burnside's lemma, after Cauchy, Frobenius) $\langle \pi_X, 1\rangle =$ number of orbits of G on X.

Proof. If $X = X_1 \cup ... \cup X_l$ disjoint union of orbits, then $\pi_X = \pi_{X_1} + ... + \pi_{X_l}$, with π_{X_j} permutation character of G on X_j, so to prove the claim it's enough to show that if G is transitive on X then $\langle \pi_X, 1\rangle = 1$. Assume G is transitive on X. Now

$$\begin{aligned}\langle \pi_X, 1\rangle &= \frac{1}{|G|}\sum_g \pi_X(g) = \frac{1}{|G}|\{(g,x) \in G \times X : gx = x\}| \\ &= \frac{1}{|G|}\sum_{x\in X}|G_x| = \frac{1}{|G|}|X||G_x| = \frac{1}{|G|}|G| = 1\end{aligned}$$

(Note the use of orbit-stabilizer theorem). □

Lemma. (7.4)
Let G act on the sets X_1, X_2. Then G acts on $X_1 \times X_2$ via $g(x_1, x_2) = (gx_1, gx_2)$. The character $\pi_{X_1\times X_2} = \pi_{X_1}\pi_{X_2}$ and so $\langle \pi_{X_1}, \pi_{X_2}\rangle =$ number of orbits of G on $X_1 \times X_2$.

Proof. If $g \in G$ then $\pi_{X_1\times X_2}(g) = \pi_{X_1}(g)\pi_{X_2}(g)$. And we have

$\langle \pi_{X_1}, \pi_{X_2}\rangle = \langle \pi_{X_1}\pi_{X_2}, 1\rangle = \langle \pi_{X_1\times X_2}, 1\rangle =$ (7.3) number of orbits of G on $X_1 \times X_2$.

□

Definition. (7.5)
Let G act on X, $|X| > 2$. Then G is *2-transitive* on X if G has precisely two orbits on $X \times X : \{(x, x) : x \in X\}$ and $\{x_1, x_2) : x_i \in X, x_1 \neq x_2\}$.

Lemma. (7.6)
Let G act on X, $|X| > 2$. Then $\pi_X = 1 + \chi$ with χ irreducible $\iff$ G is 2-transitive on X.

Proof. $\pi_X = m_1 1 + m_2\chi_2 + ... + m_l\chi_l$ with $1, \chi_2, ..., \chi_l$ distinct irreducible characters and $m_i \in \mathbb{N}$. Then

$$\langle \pi_X, \pi_X \rangle = \sum_{i=1}^{l} m_i^2$$

hence G is 2-transitive on X $\iff$ $l = 2, m_1 = m_2 = 1$. □

Example. (7.7)
Consider S_n acting on $X = \{1, ..., n\}$ which is 2-transitive. Hence $\pi_X = 1 + \chi$ with χ irreducible of degree $n - 1$. Similarly for A_n $(n > 3)$.

Example. (7.8)
Consider $G = S_4$.

ccl	1	3	8	6	6
rep	1	(12)(34)	(123)	(1234)	(12)
$1_G = \chi_1$	1	1	1	1	1
sign $= \chi_2$	1	1	1	-1	-1
$\pi_X - 1 = \chi_3$	3	-1	0	-1	1
$\chi_2\chi_3 = \chi_4$	3	-1	0	1	-1
χ_5	2	...	...	...	...

← Get by column orthogonality

Last lecture we were talking about using column orthogonality to find χ_5. Indeed we have

$$\chi_{reg} = \chi_1 + \chi_2 + 3\chi_3 + 3\chi_4 + 2\chi_5$$

So we can use this to find χ_5. Also, $S_4/V_4 \cong S_3$ by 'lifting' – see next chapter.

7.1 Alternating groups

Suppose $g \in A_n$. In 1A we've known that $|\mathcal{C}_{S_n}(g)| = |S_n : C_{S_n}(g)|$ and $|\mathcal{C}_{A_n}(g)| = |A_n : C_{A_n}(g)|$.

These are not necessarily equal. For example, $\sigma = (123) \in A_3$, $\mathcal{A}_3(\sigma) = \{\sigma\}$, but $\mathcal{S}_\ni(\sigma) = \{\sigma, \sigma^{-1}\}$.

Lemma. (7.9)
Let $g \in A_n$. Then if g commutes with some odd permutation in S_n then $\mathcal{C}_{S_n}(g) = \mathcal{C}_{A_n}(g)$; otherwise $\mathcal{C}_{S_n}(g)$ splits into two ccls in A_n of equal size.

For example, consider $G = A_4$, so $|G| = 12$.

	1	$(12)(34)$	(123)	$(123)^{-1}$
$1_G = \chi_1$	1	1	1	1
$\pi_X - 1_G = \chi_2$	3	-1	0	0
χ_3	1	1	ω	ω^2
χ_4	1	1	ω^2	ω

Note that if we ignore the second row and first column, the table becomes identical to that of $C_3 \cong G/V_4$. This is not a coincident, and is actually called *lifting*.

8 Normal subgroups and lifting characters

Lemma. (8.1)
Let $N \triangleleft G$. Let $\tilde{\rho} : G/N \to GL(V)$ be a representation of G/N. Then

$$\begin{aligned} \rho : G &\xrightarrow{canonical} G/N \quad \xrightarrow{\tilde{\rho}} GL(V) \\ g &\to \qquad\qquad\qquad \tilde{\rho}(gN) \end{aligned}$$

is a representation of G, where $\rho(g) := \tilde{\rho}(gN)$. Moreover, ρ is irreducible iff $\tilde{\rho}$ is irreducible.

The corresponding characters satisfy $\chi(g) = \tilde{\chi}(gN)$. We say that $\tilde{\chi}$ *lifts* to χ. The lifting $\tilde{\chi} \to \chi$ is a bijection between irreducible representations of G/N and irreducible representations of G with N in ker.

Well this looks like Q4/Q12 in the first example sheet.

Proof. Note $\chi(g) = \mathrm{tr}(\rho(g)) = \mathrm{tr}(\tilde{\rho}(gN)) = \tilde{\chi}(gN) \forall g$, and $\chi(1) = \tilde{\chi}(N)$. SO have some degree (?).

Bijection: if $\tilde{\chi}$ is a charcter of G/N-representation and χ is its lift to G, then $\chi(N) = \chi(1)$. Also, if $k \in N$ then

$$\chi(k) = \tilde{\chi}(kN) = \tilde{\chi}(N) = \chi(1)$$

So $N \leq \ker \chi$.

Now let χ be character of G with $N \leq \ker \chi$. Suppose $\rho : G \to GL(V)$ affords χ. Define

$$\begin{aligned} \tilde{\rho} : &G/N \quad \to GL(V) \\ &gN \qquad \to \rho(g) \end{aligned}$$

Check this is well-defined (uses $N \leq \ker \chi$) and $\tilde{\rho}$ is homomorphism, hence gives representation of G/N. If $\tilde{\chi}$ is the character of $\tilde{\rho}$ then $\tilde{\chi}(gN) = \chi(g)\ \forall g \in G$. So $\tilde{\chi}$ lifts to χ.
Check irreducibility. □

Lemma. (8.2)
The derived subgroup, $G' = \langle [a,b], a, b \in G \rangle$ of G is the unique minimal normal subgroup of G s.t. G/G' is abelian, i.e. G/N is abelian $\implies G' \leq N$ and $G^{ab} = G/G'$ is abelian, where G^{ab} is the *abelianisation* of G.
G has precisely $l = |G/G'|$ representations of $\dim 1$, all with kernel containing G' and obtained by lifting from G/G'. In particular, $l||G|$.

Proof. $G' \triangleleft G$ is an easy exercise.

Let $N \triangleleft G$. Let $h, g \in G$, so

$$\begin{aligned} g^{-1}h^{-1}gh \in N \iff& \qquad\qquad (gh)N = (hg)N \\ [g,h] \iff (gN)(hN) &= (hN)(gN) \end{aligned}$$

So $G' \leq N \iff G/N$ is abelian. Since $G' \lhd G$ we deduce G/G' is abelian.

By (4.5), G/G' has exactly l irreducible characters $\tilde{\chi}_1, ..., \tilde{\chi}_l$ all of degree 1. The lifts of these to G also have degree 1 and by (8.1) these are precisely the irreducible characters χ_i of G s.t. $G' \leq \ker \chi_i$. But any linear character of G is a homomorphism $\chi : G \to \mathbb{C}^*$, hence $G' \leq \ker \chi$ ($\chi(ghg^{-1}h^{-1}) = \chi(g)\chi(h)\chi(g^{-1}\chi(h)^{-1} = 1$), so the $\chi_1, ..., \chi_l$ are all the linear characters of G. □

Examples:
(a) If $G = S_n$, show $s'_n = A_n$. Thus since $G/G' \cong C_2$, S_n must have exactly two linear characters.
(b) Consider $G = A_4$. We've seen previously that this can be lifted from C_3 using (8.1),(8.2).

Lemma. (8.4)
G is not simple iff $\chi(g) = \chi(1)$ for some irreducible character $\chi \neq 1_G$ and some $1 \neq g \in G$.
Any normal subgroup of G is the intersection of the kernels of some of the irreducible characters of G:

$$N = \bigcap_i \ker \chi_i$$

Proof. If $\chi(g) = \chi(1)$ for some non-trivial irreducible character χ (afforded by ρ, say). Then $g \in \ker \rho$ (5.3), so if $g \neq 1$, then $1 \neq \ker \rho \overset{\lhd}{\neq} G$.
If $1 \neq N \overset{\lhd}{\neq} G$, take irreducible $\tilde{\chi}$ of G/N, $\tilde{\chi}$ non-trivial. Lift to get an irreducible χ, afforded by ρ of G, then $N \leq \ker \rho \lhd G$. So $\chi(g)\ chi(1)$ for $g \in N$.
We claim that, if $1 \neq N \lhd G$, then N is the intersection of the kernels of the lifts of all the irreducibles of G/N.
$\leq$ is clear from (8.1). If $g \in G \setminus N$, then $gN \neq N$. so $\tilde{\chi}(gN) \neq \tilde{\chi}(N)$ for some irreducible $\tilde{\chi}$ of G/N. Lifting $\tilde{\chi}$ to χ, we have $\chi(g) \neq \chi(1)$. □

Recall $\ker \chi = \{g \in G : \chi(g) = \chi(1)\}$. (5.3) : $g \in \ker \chi \iff g \in \ker \rho$.

9 Dual spaces and tensor products of representations

Recall (5.5):
• $\mathcal{C}(G)$ is $\mathbb{C}$-space of class functions on G;
• endowed with irreducible product, $\dim \mathcal{C}(G) = k$, orthonormal basis of irreducible characters of G (5.6)l
• there exists an involution (ring homomorphism of order 2): $f \to f^*$ where $f^*(g) = f(g^{-1})$.

Lemma. (9.1)
Let $\rho : G \to GL(V)$, representation over F, and let $V^* = Hom_F(V, F)$, dual space of V. Then V^* is a G-space under

$$(\rho^*(g)\phi)(v) = \phi(\rho(g^{-1})v)$$

called the *dual representation* to ρ. Its charcater is $\chi_{\rho^*}(g) = \chi_\rho(g^{-1})$.

Proof.

$$\begin{aligned}\rho^*(g_1)(\rho^*(g_2)\phi)(v) &= (\rho^*(g_2)\phi)(\rho(g_1^{-1})(v)) \\ &= \phi(\rho(g_2^{-1})\rho(g_1^{-1})v) \\ &= \phi(\rho(g_1g_2)^{-1}(v)) \\ &= (\rho^*(g_1g_2)\phi)(v)\end{aligned}$$

So this is a representation. For its character, fix $g \in G$ and let $e_1, ..., e_n$ be basis of V of eigenvectors of $\rho(g)$, say $\rho(g)e_j = \lambda_j e_j$. Let $\varepsilon_1, ..., \varepsilon_n$ be dual basis. We claim that $\rho^*(g)\varepsilon_j = \lambda_j^{-1}\varepsilon_j$:

$$(\rho^*(g)\varepsilon_j)(e_i) = \varepsilon_j(\rho(g^{-1})e_i) = \varepsilon_j\lambda_i^{-1}e_i = \lambda_j^{-1}\varepsilon_j e_i \forall i$$

So $\chi_{\rho^*}(g) = \sum \lambda_j^{-1} = \chi_\rho(g^{-1})$. □

Definition. (9.2)
$\rho : G \to GL(V)$ is <u>*self-dual*</u> if $V \cong V^*$ (as G-spaces). Over $\mathbb{C}$, this holds iff $\chi_\rho(g) = \chi_\rho(g^{-1})$ $(= \overline{\chi_\rho(g)})$ $\forall g$, iff $\chi_\rho(g) \in \mathbb{R}$ for all g.

Exercise: all irreducible representations of S_n are self-dual (the ccls are determined by cycle type, so g, g^{-1} are always S_n-conjugate. Not always true for A_n.

9.1 tensor products

Let V, W be $F-$spaces, $\dim V = m$, $\dim W = n$. Fix bases $v_1, ..., v_m$ and $w_1, ..., w_n$ of V, W respectively. The *tensor product space* $V \otimes_F W$ is an *nm*-dimensional F-space with basis $\{v_i \otimes w_j : 1 \le i \le m, 1 \le j \le n\}$. Thus
(a) $V \otimes W = \{\sum_{i,j} \lambda_{ij} v_i \otimes w_j : \lambda_{ij} \in F\}$ with 'obvious' addition and scalar multiplication;
(b) If $v = \sum_i \alpha_i v_i \in V$, $w = \sum_j \beta_j w_j \in W$, define $v \otimes w := \sum_{i,j} \alpha_i\beta_j(v_i \otimes w_j)$.

Remark. Not all elements of $V \otimes W$ are of this form: some are combinations, e.g. $v_1 \otimes w_1 + v_2 \times w - 2$, which can't be further simplified. (like entangled)

Lemma. (9.3)
(i) For $v \in V$, $w \in W$, $\lambda \in F$, $(\lambda v) \otimes w = \lambda(v \otimes w) = v \otimes (\lambda w)$;
(i) If $x_1, x_2, x \in V$, $y_1, y_2, y \in W$, then

$$(x_1 + x_2) \otimes y = (x_1 \otimes y) + (x_2 \otimes y),$$
$$x \otimes (y_1 + y_2) = (x \otimes y_1) + (x \otimes y_2)$$

Proof. (i) $v = \sum \alpha_i v_i$, $w = \sum \beta_j w_j$. Then just multiply out everything we get the desired equality. (ii) is similar. □

Lemma. (9.4)
If $\{e_1, ..., e_m\}$ is a basis of V, $\{f_1, ..., f_n\}$ is a basis of W, then $\{e_i \otimes f_j : 1 \le i \le m, 1 \le j \le n\}$ is a basis of $V \otimes W$.

Proof. Writing $v_k = \sum_i \alpha_{ik} e_i$, $w_l = \sum_j \beta_{jl} f_j$, we have

$$v_k \otimes w_l = \sum \alpha_{ik} \beta_{jl} e_i \otimes f_j$$

Hence $\{e_i \otimes f_j\}$ spans $V \otimes W$ and, since we have nm of them, they form a basis. □

Remark. One can define $V \otimes W$ in a basis-independent way in the first place, see Teleman chapter 6.

Proposition. (9.5)
Let $\rho : G \to GL(V)$, $\rho' : G \to GL(V')$ be representations of G. Define $\rho \otimes \rho' : G \to GL(V \otimes V')$ by

$$(\rho \otimes \rho')(g) : \sum \lambda_{ij} v_i \otimes w_j \to \sum \lambda_{ij} \rho(g) v_i \otimes \rho'(g) w_j$$

Then $\rho \otimes \rho'$ is a representation of G with character

$$\chi_{\rho \otimes \rho'}(g) = \chi_\rho(g) \chi_{\rho'}(g) \forall g \in G$$

Hence product of two characters of G is still a character of G.

Proof. On Tuesday. □

(After lecture 11: this is the first notes to get beyond 1000 lines!)

Remark. (9.6)
Sheet 1, Q2 says ρ irreducible, ρ' of degree 1, then $\rho \otimes \rho'$ irreducible; if ρ' is not of deg 1 this is usually false.

Proof. (of 9.5)
It's clear that $(\rho\otimes\rho')(g) \in GL(V\otimes V')\ \forall g \in G$ and so $\rho\otimes\rho'$ is a homomorphism $G \to GL(V \otimes V')$. Let $g \in G$. Let $V_1, ..., v_m$ be basis of V of eigenvectors of $\rho(g)$; let $w_1, ..., w_n$ be a basis of V'. Say:

$$\rho(g)v_j = \lambda_j v_j, \rho'(g)w_j = \mu_j w_j$$

Then

$$\begin{aligned}(\rho \otimes \rho')(g)(v_i \otimes w_j) &= \rho(g)v_i \otimes \rho'(g)w_j \\ &= \lambda_i v_i \otimes \mu_j w_j \\ &= (\lambda_i\mu_j)(v_i \otimes w_j)\end{aligned}$$

So $\chi_{\rho\otimes\rho'}(g) = \sum_{i,j} \lambda_i\mu_j = (\sum \lambda_i)(\sum \lambda_j) = \chi_\rho(g)\chi_{\rho'}(g)$ □

Now work over $\mathbb{C}$. Take $V = V'$ and define $V^{\otimes 2} = V \otimes V$.
Let

$$\tau : \sum \lambda_{ij} v_i \otimes v_j \to \sum \lambda_{ij}\lambda_j \otimes v_i$$

which is a linear G-endomorphism of $V^{\otimes 2}$, s.t. $\tau^2 = 1$ (so eigenvalues ± 1).

Definition. (9.7)

$$\begin{aligned}S^2V &= \{v \in V^{\otimes 2} : \tau(x) = x\}, \\ \wedge^2 V &= \{x \in V^{\otimes 2} : \tau(x) = -x\}\end{aligned}$$

known as the *symmetric square* of V and *exterior square* of V respectively.

Lemma. (9.8)
S^2V and $\wedge^2V$ are G-subspaces of $V^{\otimes 2}$ and $V^{\otimes 2} \cong S^2V \otimes \wedge^2 V$. S^2V has basis $\{v_iv_j := v_i \otimes v_j + v_j \otimes v_i : 1 \leq i \leq j \leq n\}$, and $\wedge^2V$ has basis $\{v_i \wedge v_j := v_i \otimes v_j - v_j \otimes v_i : 1 \leq i < j \leq n\}$. Hence we have $\dim S^2V = \frac{1}{2}n(n+1)$ and $\dim \wedge^2 V = \frac{1}{2}n(n-1)$.

Proof. Exercise in linear algebra.
To show $V^{\otimes 2}$ is reducible, write $x \in V^{\otimes 2}$ as $x = \frac{1}{2}(x + \tau(x)) + \frac{1}{2}(x - \tau(x))$, which is in S^2V and $\wedge^2V$ respectively. □

In fact, $V^{\otimes 2}$, $V^{\otimes 3} = V \otimes V \otimes V$, ...,etc. are never irreducible if $\dim V > 1$.

Lemma. (9.9)
If $\rho : G \to GL(V)$ is a representation affording character χ, then $\chi^2 = \chi_S + \chi_\wedge$ where χ_s ($= S^2\chi$) is the character of G in the subrepresentation S^2V, and $\chi_\wedge$ ($= \wedge^2\chi$) is the character of G in the subrepresentation $\wedge^2V$. Moreover, for $g \in G$,

$$\chi_s(g) = \frac{1}{2}(chi^2(g) + \chi(g^2)), \chi_\wedge(g) = \frac{1}{2}(\chi^2(g) - \chi(g^2)).$$

Proof. Let's compute the characters $\chi_s, \chi_\wedge$. Fix $g \in G$. Let $v_1, ..., v_n$ be a basis of eigenvectors of $\rho(g)$, say $\rho(g)v_i = \lambda_i v_i$ (we drop the ρ to write $gv_i = \lambda_i v_i$ for simplicity below). Then

$$\begin{aligned}gv_iv_j &= \lambda_i\lambda_j v_iv_j \\ gv_i \wedge v_j &= \lambda_i\lambda_j v_i \wedge v_j\end{aligned}$$

Hence $\chi_s(g) = \sum_{1\le i\le j\le n} \lambda_i\lambda_j$ and $\chi_\wedge(g) = \sum_{1\le i<j\le n} \lambda_i\lambda_j$. Now,

$$\begin{aligned}(\chi(g))^2 &= (\sum \lambda_i)^2 \\ &= \sum \lambda_i^2 + 2\sum_{i<j} \lambda_i\lambda_j \\ &= \chi(g^2) + 2\sum_{i<j} \lambda_i\lambda_j \\ &= \chi(g^2) + 2\chi_\wedge(g)\end{aligned}$$

So $\chi_\wedge(g) = \frac{1}{2}(\chi^2(g) - \chi(g^2))$. But $\chi^2 = \chi_s + \chi_\wedge$ so we get the expression for $\chi_s(g)$. □

Example. (9.10)
Consider our usual example $G = S_4$ (see 7.8).

	1	(12)(34)	(123)	(12)	(1234)
1_G	1	1	1	1	1
sign	1	1	1	−1	−1
$\chi_3 = \pi_X - 1$	3	−1	0	1	−1
$\overline{\chi}_3$	3	−1	0	−1	1
χ_2	2	2	−1	0	0

χ_3^2	9	1	0	1	1
$\chi_3(g^2)$	3	3	0	3	-1
$S^2\chi_3$	6	2	0	2	0
$\Lambda^2\chi_3$	3	-1	0	-1	1

Notice that $\wedge^2\chi_3 = \bar{\chi}_3$ (irreducible since $\langle \chi_\wedge, \chi_\wedge \rangle = 1$),
$S^2\chi_3 = 1 + \chi_3 + \chi 5$: The inner product is 3 and it contains 1 ,χ_3, so the one left is χ_5.

Characters of $G \times H$ (seen in (4.5) for abelian groups):

Proposition. (9.11)
If G, H are finite groups with irreducible characters $\chi_1, ..., \chi_k$ and $\psi_1, ..., \psi_r$ respectively, then the irreducible characters of the direct product $G \times H$ are precisely $\{\chi_i\psi_j : 1 \leq i \leq k, 1 \leq j \leq r\}$, where $\chi_i\psi_j(g, h) = \chi_i * g(\psi_j(h)$.

Proof. If $\rho : G \to GL(V)$, $\rho' : H \to GL(W)$ affording χ and ψ respectively, then

$$\begin{aligned} \rho \otimes \rho' : G \times H \to \quad & GL(V \otimes W) \\ (g, h) \to \quad & \rho(g) \otimes \rho'(h) \quad v_i \otimes w_j \to \rho(g)v_i \otimes \rho'(h)w_j \end{aligned}$$

is a representation of $G \times H$ on $V \otimes W$ by (9.5), and $\chi_{\rho\otimes\rho'} = \chi\psi$, again by (9.5). We claim that $\chi_i\psi_j$ are distinct and irreducible:

$$\begin{aligned} \langle \chi_i\psi_j, \chi_r\psi_s \rangle_{G\times H} &= \frac{1}{|G \times H|} \sum_{(g,h)} \overline{\chi_i\psi_j(g, h)}\chi_r\psi_s(g, h) \\ &= (\frac{1}{|G|}\overline{\chi_i(g)}\chi_r(g))(\frac{1}{|H|}\sum_h \overline{\psi_j(h)}\psi_s(h)) \\ &= \delta_{ir}\delta_{js} \end{aligned}$$

...tbc.
Let's complete on $\chi_i\psi_j$ being distinct and irreducible:
Complete set: $\sum_{i,j}(\chi_i\psi_j)(1)^2 = \sum_i \chi_i(1)^2 \sum_j \psi_j(1)^2 = |G||H| = |G \times H|$ □

9.2 Symmetric and extreior powers

Let V be a vector space, $\dim_F V = d$, with basis $\{v_1, ..., v_d\}$. Let $V^{\otimes n} = V \otimes ... \otimes V$, with basis $\{v_{i_1} \otimes ... \otimes v_{i_n} : (i_1, ..., i_n) \in \{1, ..., d\}^n\}$, so $\dim V^{\otimes n} = d^n$.

S_n-action: for any $\sigma \in S_n$, we can define linear map

$$\begin{aligned} \sigma : V^{\otimes n} \to & \quad V^{\otimes n} \\ v_1 \otimes ... \otimes v_n \to & v_{\sigma^{-1}(1)} \otimes ... \otimes v_{\sigma^{-1}(n)} \end{aligned}$$

for $v_1, ..., v_n \in V$, permuting posutums of vectors in a tensor.

For example, $(12)(v_1 \otimes v_2 \otimes v_3) = v_2 \otimes v_1 \otimes v_3$, $(13)(v_2 \otimes v_1 \otimes v_3) = v_3 \otimes v_1 \otimes v_2$.

Check that this defines a representation of S_n on $V^{\otimes n}$ (extended linearly).

G-action: given representation $\rho : G \to GL(V)$, then the action of G on $V^{\otimes n}$ is

$$\rho^{\otimes n}(g) : v_1 \otimes ... \otimes v_n = \rho(g)v_1 \otimes ... \otimes \rho(g)v_n$$

extended linearly, and this commutes with the S_n-action. We can decompose $V^{\otimes n}$ as S_n-module, and each isotypical component (4.?) is G-invariant subspace of $V^{\otimes n}$. In particular:

Definition. (9.12)
For G-space V, define
(i) the nth symmetric power of V, $S^n V = \{x \in V^{\otimes n} : \sigma(x) = x \forall \sigma \in S_n\}$;
(ii) the nth exterior power of V, $\wedge^n V = \{x \in V^{\otimes n} : \sigma(x) = sign(\sigma) x \forall \sigma \in S_n\}$.
Both are G-subspaces of $V^{\otimes n}$, but for $n > 2$, $S^n V \oplus \wedge^n V \lneq V^{\otimes n}$, so in general there are lots of others for the S_n-action.

(9.13) See Sheet 3 Q7 for bases of $S^n V$, $\wedge^n V$ and their characters.

9.3 Tensor algebra

Take $char F = 0$.

Definition. (9.14)
Let $T^n V = V^{\otimes n}$. The tensor algebra of V is $TV := \oplus_{n \geq 0} T^n V$, $T^0 V = F$.
This is F-space and is a (non-commutative) graded ring with product $x \in T^n V$, $y \in T^m V$, $x \cdot y = x \otimes y \in T^{n+m} V$.
There are two graded quotient rings

$$\begin{aligned} SV = & TV/(\text{ideal generated by all } U \otimes V - V \otimes U) \\ \wedge V = & TV/ \text{ ideal generated by all } V \otimes V \end{aligned}$$

called the symmetric algebra and exterior algebra respectively.

Definition. (9.15)
The 2-submodule of $\mathcal{C}(G)$ spanned by irreducible characters of G is the character

ring of G, $R(G)$. Elements of $R(G)$ are called generalised/virtual characters if $\psi = \sum n_\chi \chi$, $n_\chi \in \mathbb{Z}$ correspondingly.
• $R(G)$ is a commutative ring and any generalised character is a difference of two characters, $\psi = \alpha - \beta$:
$\alpha = \sum_{n_\chi \geq 0} n_\chi \chi, \beta = -\sum_{n_\chi < 0} n_\chi \chi$.
The $\{\chi_i\}$ form a $\mathbb{Z}$-basis for $R(G)$ as a free $\mathbb{Z}$-module.
• Suppose ψ is virtual character and $\langle \psi, \psi \rangle = 1$ and $\psi(1) > 0$. Then ψ is actually the character of an irreducible representation of G.
List irreducible characters of G: $\chi_1, ..., \chi_k$, $\psi = \sum n_i \chi_i$; orthonormality says $\langle \psi, \psi \rangle = \sum n_i^2$, so $\sum n_i^2 = 1$, meaning $n_i = \pm 1$ for exactly one i and $n_j = 0$ for $j \neq i$. Since $\psi(1) > 0$, we must have $n_i = +1$.
• Henceforth we don't distinguish between a character and its negative and we often study generalised characters of norm 1 rather than irreducible characters.

10 Restriction and induction

Throughout we set $H \leq G$, $F = \mathbb{C}$.

Definition. (10.1, restriction)
Let $\rho : G \to GL(V)$ be representation affording χ. We can think of V as a H-space by restricting attention to $h \in H$. We then get

$$Res_H^G \rho : H \to \quad GL(V)$$

This is sometimes written as ρ_H or $\rho \downarrow_H$, the restriction of ρ to H. It affords the character $Res_H^G \chi = \chi_H = \chi \downarrow_H$.

Lemma. (10.2)
If ψ is any non-zero character of $H \leq G$, then there exists irreducible charcater χ of G s.t. $\langle Res_H^G \chi, \psi \rangle_H \neq 0$. We say ψ is a constituent of $Res_H^G \chi$.

Proof.

$$0 \neq \frac{|G|}{|H|}\psi(1) = \langle \pi_{reg} \downarrow_H, \psi \rangle = \sum_1^k \deg \chi_i \langle \chi_i \downarrow_H, \psi \rangle$$

where ψ_i are irreducible characters of G. □

Lemma. (10.3)
Let χ be irreducible character of G, and let $Res_H^G \chi = \sum c_I \chi_i$ with χ_i irreducible characters of H, $c_i \in \mathbb{Z}_{\geq 0}$. Then

$$\sum c_I^2 \leq |G : H|$$

with equality iff $\chi(g) = 0 \ \forall g \in G \setminus H$.

Proof.

$$\sum c_i^2 = \langle Res_H^G \chi, Res_H^G \chi \rangle_H = \frac{1}{|H|} \sum_{h \in H} |\chi(h)|^2$$

But

$$\begin{aligned} 1 = \langle \chi, \chi \rangle G &= \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2 \\ &= \frac{1}{|G|} (\sum_{h \in H} |\chi(h)|^2 + \sum_{g \in G \setminus H} |\chi(g)|^2) \\ &= \frac{|H|}{|G|} \sum c_i^2 + \underbrace{\frac{1}{|G|} \sum_{g \in G \setminus H} |\chi(g)|^2}_{\geq 0} \end{aligned}$$

So $\sum c_i^2 \leq |G : H|$, with equality holds iff $\chi(g) = 0$. $\forall g \in G \setminus H$. □

Example. Let $G = S_5$, $H = A_5$. This has 7 representations of degree $1, 1, 4, 4, 5, 5, 6$ respectively, where if we restrict to H, the two representations of degree $1, 4, 5$ combines into one of the same degree respectively; however, the degree 6 representation splits into two irreducible representations of degree 3. In the first case we have $\chi(g) \neq 0$ somewhere outside H; for the degree 6 representation, $\chi(g) = 0\ \forall g \in S_5 \setminus A_5$. All restrictions are irreducible if $|G : H| = 2$ which is the case here. Fact: $\chi \downarrow_H$ all constituents have same degree if $H \triangleleft G$ (Janes-Liebeck, chapter 20).

Let's talk about induced characters.

Definition. (10.4)
If $\psi \in \mathcal{C}(H)$, define $Ind_H^G \psi(g) = \frac{1}{|G|} \sum_{\chi \in G} \mathring{\psi}(x^{-1}gx)$, where

$$\mathring{\psi}(g) = \begin{cases} \psi(g) & g \in H \\ 0 & g \notin H \end{cases}$$

We also write $Ind_H^G \psi(g)$ as $\psi \uparrow^G = \psi^G$.

Lemma. (10.5)
If $\psi \in \mathcal{C}(H)$ then $Ind_H^G \psi \in \mathcal{C}(G)$ and $Ind_H^G \psi(1) = |G : H| \psi(1)$.

Proof. This is clear, noting that $Ind_H^G \psi(1) = \frac{1}{H} \sum \mathring{\psi}(1) = |G : H| \psi(1)$. □

Let $n = |G : H|$. Let $1 = t_1, t_2, ..., t_n$ be a *left transversal* of H in G (complete set of coset representatives), so that $t_1 H = H$, $t_2 H, ..., t_n H$ are precisely the n left cosets of H in G.

Lemma. (10.6)
Given left transversal as above,

$$Ind_H^G \psi(g) = \sum_{i=1}^{n} \mathring{\psi}(t_i^{-1} g t_i)$$

Proof. For $h \in H$, $\mathring{\psi}((t_i h)^{-1} g (t_i h)) = \mathring{\psi}(t_i^{-1} g t_i)$ as ψ is a class function on H. □

Theorem. (10.7, Frobenius reciprocity)
$H \leq G$. ψ is a class function for H, ϕ is a class function for G. Then

$$\langle \underbrace{Res_H^G \phi}_{in\ \mathcal{C}(H)}, \psi \rangle_H = \langle \phi, \underbrace{Ind_H^G \psi}_{in\ \mathcal{C}(G)} \rangle_G$$

Proof. We want to show $\langle \phi_H, \psi \rangle_H = \langle \phi, \psi^G \rangle_G$:

$$\langle \phi, \psi^G \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\phi(g)} \psi^G(g) = \frac{1}{|G||H|} \sum_{g, x \in G} \overline{\phi(g)} \mathring{\psi}(x^{-1} g x)$$

Put $y = x^{-1} g x$. The above then equals

$$\frac{1}{|G||H|} \sum_{x, y \in G} \overline{\phi(y)} \mathring{\psi}(y) = \frac{1}{|H|} \sum_{y \in G} \overline{\phi(y)} \mathring{\psi}(y)$$

which is independent of x, and then equals

$$\frac{1}{|H|}\sum_{y\in H}\overline{\phi(y)}\psi(y) = \langle \phi_H, \psi\rangle_H$$

□

Corollary. (10.8)
If ψ is a character of H, then $Ind_H^G\psi$ is a character of G.

Proof. Let χ be an irreducible character of G. Then

$$\langle Ind_H^G\psi, \chi\rangle = \langle \psi, Res_H^G\chi\rangle \in \mathbb{Z}_{\geq 0}$$

since ψ and $Res_H^G\chi$ are characters. Hence $Ind_H^G\psi$ is a linear combination of irreducible charcaters with non-negative coefficients, hence a character. □

Lemma. (10.9)
Let ψ be a character of $H \leq G$, and let $g \in G$. Let

$$\mathcal{C}_G(g) \cup H = \bigcup_{i=1}^{m} \mathcal{C}_H(x_i)$$

(disjoint union), where the x_i are representatives of the H-ccls of elements of H conjugate to g.
If $m = 0$, then $Ind_H^G\psi(g) = 0$. Otherwise

$$Ind_H^G\psi(g) = |C_G(g)| \cdot \sum_{i=1}^{m} \frac{\psi(x_i)}{|C_H(x_i)|}$$

Proof. Assume $m > 0$. Let $X_i = \{x \in G : x^{-1}gx \in H$ and is conjugate in H to $x_i\}$ $\forall 1 \leq i \leq m$. The X_i are pairwise disjoint, and their union is $\{x \in G : x^{-1}gx \in H\}$. By definition,

$$\begin{aligned}
Ind_H^G\psi(g) &= \frac{1}{|H|}\sum_{\alpha\in G}\mathring{\psi}(x^{-1}gx)\\
&= \frac{1}{|H|}\sum_{i=1}^{m}\sum_{x\in X_i}\psi(x^{-1}gx)\\
&= \frac{1}{|H|}\sum_{i=1}^{m}\sum_{x\in X_i}\psi(x_i)\\
&= \sum_{i=1}^{m}\frac{|X_i|}{|H|}\psi(x_i)
\end{aligned}$$

and evaluate $\frac{|X_i|}{|H|}$ to get what we want... although a bit tedious: Fix $1 \leq i \leq m$ and choose some $g_i \in G$ s.t. $g_i^{-1}gg_i = x_i$ so $\forall c \in C_G(g)$ and $h \in H$,

$$\begin{aligned}
(cg_ih)^{-1}g(cg_ih) &= h^{-1}g_i^{-1}c^{-1}gcg_ih\\
&= h^{-1}g_i^{-1}c^{-1}cgg_ih\\
&= h^{-1}g_i^{-1}gg_ih\\
&= h^{-1}x_ih \in H
\end{aligned}$$

i.e. $cg_ih \in X_i$, hence $C_G(g)g_iH \subseteq X_i$;

Convserly, if $x \in X_i$ then $x^{-1}gx = h^{-1}x_ih = h^{-1}(g_i^{-1}gg_i)h$ for some $h \in H$; thus $xh^{-1}g_i^{-1} \in C_G(g)$. So $x \in C_G(g)g_ih \subseteq C_G(g)g_iH$. Conclude $X_i = C_G(g)g_iH$, thus

$$|X_i| = |C_G(g)g_iH| = \frac{|C_G(g)||H|}{|H \cap g_i^{-1}C_G(g)g_i}$$

(see notes at end). Finally $g_i^{-1}C_G(g)g_i = C_G(g_i^{-1}gg_i) = C_G(x_i)$. Thus

$$\begin{aligned}|X_i| &= |H : H \cup C_G(x_i)||C_G(g)| \\ &= |H : C_H(x_i)||C_G(g)|\end{aligned}$$

Thus,

$$\begin{aligned}\frac{|X_i|}{|H|} &= \frac{|H : C_H(x_i)||C_G(g)|}{|H|} \\ &= \frac{|C_G(g)|}{|C_H(x_i)|}\end{aligned}$$

for each $1 \leq i \leq m$. □

Note: if $H, K \leq G$, a double coset of H and K in G is a set $HgK = \{hgk : h \in H, k \in K\}$ for some $g \in G$.

Facts:
- two double cosets are either disjoint or equal;
- $|HgK| = \frac{|H||K|}{|H \cap gKg^{-1}} = \frac{|H||K|}{|g^{-1}Hg \cap K|}$ (prove this: it's a bit like $|HK|$).

Example. Consider $H = C_4 = \langle (1234\rangle \leq G = S_4$, of index 6. Char of induced representation $Ind_H^G(\alpha)$ wher $|alpha$ is faithful 1-dim representation of C_4. If $\alpha((1234)) = i$, then char of α is $(1\ i\ -1\ i)$ for $(1), (1234), (13)(24), (1432)$. The induced representation of S_4, we know $Ind_{C_4}^{S_4}\chi_\alpha$ evaluates to 6 at (1) (by (10.5)) and to 0 at (12) and (123).
For (12)(34) only one of the three elements of S_4 it's conjugate to, lies in H, namely (13)(24). So $Ind_H^G\chi_\alpha((12)(34)) = 8(-1/4) = -2$.
For (1234), it is conjugate to 6 elements of S_4 of which two are in C_4, namely (1234) and (1432). So $Ind_H^G\chi_\alpha(1234) = 4(\frac{i}{4} - \frac{i}{4}) = 0$.

10.1 Induced representations

Let $H \leq G$, of index n. Let $1 = t_1, t_2, ..., t_n$ transversal, i.e. $H, t_2H, ..., t_nH$ are left cosets of H. Let W be a H-space.

Lemma. (10.10)
$Ind_{\{1\}}^G 1 = \rho_{reg}$.

Definition. (10.11) Let $V := W \oplus t_2 \otimes W \oplus ... \oplus t_n \otimes W = \bigoplus_{t_i} t_i \otimes W$, where $t_i \otimes W = \{t_i \otimes w : w \in W\}$. So $\dim V = n \dim W$. We write $V = Ind_H^G W$.

G-action: Let $g \in G$. $\forall i \exists$ unique j with $t_j^{-1} g t_i \in H$ (namely $t_j H$ is the coset containing gt_i). You got to understand where did this g come from, otherwise you can't make progress. Define

$$g(t_i \otimes W) = t_j \otimes ((t_j^{-1} g t_i) w)$$

We drop $\otimes$ from now. Check this is a G-action. Then

$$\begin{aligned} g_1(g_2 t_i w) &= g_1(t_j (t_j^{-1} g_2 t_i) w) \\ &= t_l((t_l^{-1} g_1 t_j)(t_j^{-1} g_2 t_i) w) \\ &= t_l(t_l^{-1}(g_1 g_2) t_i) w = (g_1)(g_2)(t_i w) \end{aligned}$$

where j and l are the unique ones such that $g_2 t_i H = t_j H$ and $g_1 t_j H = t_l H$.

It has the 'right' character: $g : t_i w \to t_j \underbrace{(t_j^{-1} g t_i)}_{} {}_{\in H} w$, so the contribution to the character is 0 unless $j = i$, i.e. if $t_i^{-1} g t_i \in H$, in which case it contributes $\psi(t_i^{-1} g t_i)$. So

$$Ind_H^G \psi_(g) = \sum_1^m \mathring{\psi}(t_i^{-1} g t_i) \text{ (10.6)}$$

Remark. (10.12)
There is Frobenius Reciprocity,

$$Hom_H(W, Res_H^G V) \cong Hom_G(Ind_H^G W, V)$$

naturally as vector spaces (W is a H-space, V is a G-space).

Lemma. (10.13)
(i) $Ind_H^G (W_1 \oplus W_2) \cong Ind_H^G W_1 \otimes Ind_H^G W_2$;
(ii) $\dim Ind_H^G W = |G : H| \dim W$.
(iii) If $H \leq K \leq G$, then $Ind_K^G Ind_H^K W \cong Ind_H^G W$.
(lecture had (10.10) here because he missed it previously, and labelled (iii) as (iv) while (10.10) as (iii)).

Proof. (10.10):

$$\begin{aligned} Ind_H^G \psi(g) &= \sum_{i=1}^n \mathring{\psi}(t_i^{-1} g t_i) \\ &= \sum_1^n \mathring{1}_H(e_i^{-1} g t_i) \\ &= |\{i : t_i^{-1} g t_i \in H\}| \\ &= |\{i : g \in t_i H t_i^{-1}\}| = |fix_X(g)| = \pi_X \end{aligned}$$

□

Remark. $\langle \psi_X, 1_G \rangle_G = \langle Ind_H^G 1_H, 1_G \rangle_G = \langle 1_H, 1_H \rangle = 1$ as predicted in chapter 7.

11 Frobenius groups

Theorem. (11.1, Frobenius theorem, 1891)
Let G be a transitive permutation group on a finite X, say $|X| = n$. Assume that each non-identity element of G fixes at most one element of X. Then

$$K = \{1\} \cup \{g \in G : g\alpha \neq \alpha \forall \alpha \in X\}$$

is a normal subgroup of G of order n.
Note that G is necessarily finite, being isomorphic to a subgroup of S_X.

Proof. (method of exceptional characters, due to M. Isaacs - chapter 7 books)
We have to show $K \triangleleft G$. Let $H = G_\alpha$ the stabiliser of $\alpha \in X$ for some $\alpha \in X$, i.e. $gG_\alpha g^{-1} = G_{g\alpha}$. Conjugates of H are stabiliers of single elements of X. No two conjugates can share a non-identity elment (by hypothesis), so H has n distinct conjugate, and G itself has $n(|H| - 1)$ elements that fix exactly one element of X. But $|G| = |X||H = n|H|$ (X and G/H are isomorphic (because transitive action) as G-sets). Hence $|K| = |G| - n(|H| - 1) = n$. Let $1 \neq h \in H$. Suppose $h = ghg^{-1}$ for some $g \in G, h' \in H$. Then h lies in both $H = G_\alpha$ and $gHg^{-1} = G_{g\alpha}$; by hypothesis $g\alpha = \alpha$, hence $g \in H$. Therefore, the ccls in G of h is precisely the ccls in H. Similarly oif $g \in C_G(h)$, then $h = ghg^{-1} \in G_{g\alpha}$ and hence $g \in H$. We conclude $C_G(h) = C_H(h)$ $(1 \neq h \in H)$. Every element of G either belongs to K or lies in one of the n stabilisers, each of which is conjugate to H. So evergy element of $G \setminus K$ is conjugate with a non-identity element of H. So $\{1, h_2, ..., h_t, y_1, ..., y_u\}$ (the representations of H-ccls and representations of ccls of G which comprise $K \setminus \{1\}$ respectively) is a set of ccls reps for G.

Take $\theta_1 = 1_G$. $\{1_H = \psi_1, ..., \psi_t\}$ be irreducible characters of H. Fix $1 \leq i \leq t$. Then, if $g \in G$, we know

$$Ind_H^G \psi_i(g) = \begin{cases} |G : H|\psi_i(1) = n\psi_i(1) & g = 1 \\ \psi_i(h_j) & g = h_j (2 \leq j \leq t) \\ 0 & g = y_k (1 \leq k \leq u) \end{cases}$$

where in the second case we appeal to $C_G(h_j) = C_H(h_j)$ and (10.9). Now fix some $2 \leq i \leq t$ and put $\theta_i = \psi_i^G - \psi_i(1)\psi_1^G + \psi_i(1)\theta_1 \in R(G)$ by (9.15). Values for $2 \leq j \leq t$, $1 \leq k\ lequ$:

	1	h_j	y_k
ψ_i^G	$n\psi_i(1)$	$\psi_i(h_j)$	0
$\psi_i'\psi_1^G$	$n\psi_i(1)$	$\psi_i(1)$	0
$\psi_i'\theta_1$	$\psi_i(1)$	$\psi_i(1)$	$\psi_i(1)$
θ_i	$\psi_i(1)$	$\psi_i(h_j)$	$\psi_i(1)$

Now calculate

$$
\begin{aligned}
\langle \theta_i, \theta_i \rangle &= \frac{1}{|G|} \sum_{g \in G} |\theta_i(g)|^2 \\
&= \frac{1}{|G|} \left(\sum_{g \in K} |\theta_i(g)|^2 + \sum_{\alpha \in X} \sum_{1 \neq g \in G_\alpha} |\theta_i(g)|^2 \right) \\
&= \frac{1}{|G|} (n\psi_i^2(1) + n \sum_{1 \neq h \in H} |\theta_i(h)|)^2 \\
&= \frac{1}{|H|} \sum_{h \in H} |\psi_i(h)|^2 \\
&= \langle \psi_i, \psi_i \rangle \\
&= 1
\end{aligned}
$$

As ψ_i is irreducible. So (by (9.15)), either θ_i or $-\theta_i$ is a character. Since $\theta_i(1) > 0$, it's $+\theta_i$, an actual character. Let $\theta = \sum_{i=1}^t \theta_i(1)\theta_i$. Column orthogonality gives $\theta(h) = \sum_{i=1}^t \psi_i(1)\psi_i(h) = 0$ $(1 \neq h \in H)$, and for any $y \in K$, $\theta(y) = \sum_{i=1}^t \psi_i^2(1) = |H|$. Hence

$$
\theta(g) = \begin{cases} |H| & g \in K \\ 0 & g \notin K \end{cases}
$$

So $K = \{g \in G : \theta(g) = \theta(1)\} \triangleleft G$. □

Definition. (11.2)
A Frobenius group is a group G having subgroup H s.t. $H \cap gHg^{-1} = 1$ $\forall g \notin H$. H is the Frobenius complement of G.

Proposition. (11.3)
Any finite Frobenius group satisfies the hypothesis of (11.1). The normal subgroup K is a Frobenius Kernel of G.

Proof. Let G be Frobenius, with complement H. Then action of G on G/H is transitive and faithful. Furthermore, if $1 \neq g \in G$ fixes both xH and yH, then $g \in xHx^{-1} \cap yhy^{-1} \implies H \cap (y^{-1}x)H(y^{-1}x)^{-1} \neq 1 \implies xH = yH$. $\square$

Example: If p, q distinct primes, $p \equiv 1 \pmod q$, the unique non-abelian group of order pq is a Frobenius group (see James-Liebeck chapter 25 or Teleman chapter 11).

Remarks:

• Thompson (thesis, 1959) proved any finite group having fixed point free automorphism of prime power order is nilpotent. This implied that in finite Frobenius group, K is nilpotent (iff K is a direct product of its sylow subgroups).

• There is no profo of (11.1) known in which character theory is not required.

12 The missing lecutre: Mackey Theory

Let's work over $\mathbb{C}$. Mackey Theory describes restriction to a subgroup $K \leq G$ of an irreducible representation $Ind_H^G W$. Here K, H are unrelated, but usually we take $K = H$, in which case we can characterise when $Ind_H^G W$ is irreducible. (?)

Special case: $W = 1_H$ (trivial H-space of dimension 1). Then $Ind_H^G W$ is the permutation representation of G on G/H (by 10.10, action on left cosets of H in G).

Recall: if G is transitive on a set X and $H = G_\alpha$ for some $\alpha \in X$, then the action of G on X is isomorphic to the action of G on G/H, namely

$$\begin{array}{ll} g \cdot \alpha \leftrightarrow & gH \ (12.1) \\ \in X & \in G/H \end{array}$$

is a well-defined bijection and commutes with G-actions ($x(g\alpha) = (xg)\alpha \leftrightarrow x(gH) = (xg)H$).

Consider the action of G on G/H and let $K \leq G$. G/H splits into K-orbits: these correspond to *double cosets* $KgH = \{KgH : k \in K, h \in H\}$, namely the K-orbit containing gH contains precisely all kgH with $k \in K$ (bunches of some gH cosets together).

Notation. (12.2)
$K \backslash G/H$ is the set of (K, H)-double cosets; they partition G. Note that $|K \backslash G/H| = \langle \pi G/K, \pi G/H \rangle$ as in (7.4). Let S be the set of representations.

Clearly $G_{gH} = gHg^{-1}$, so $K_{gH} = gHg^{-1} \cap K = Hg$.

So by (12.1), the action of K on the orbit containing gH is isomorphic to the action of K on K/Hg. From this, using $Ind_H^G 1_H = \mathbb{C}(G/H)$ and, if $X = \cup X_i$ a decomposition into orbits, then $\mathbb{C}X = \oplus_i \mathbb{C}X_i$, we get

Proposition. (12.3)
G is a finite group, $H, K \leq G$. Then

$$Res_K^G Ind_H^G 1 \cong \oplus_{g \in S} Ind_{gHg^{-1}}^K 1$$

I think this is some application:
Let $S = \{g_1 = 1, g_2, ..., g_r\}$ be s.t. $G = \cup_i Kg_iH$. Write $H_g = gHg^{-1} \cap K$ ($\leq K$). (ρ, W) is representation of H. For $g \in G$, define (ρ_g, W_g) to be the representation of Hg with the same underlying vector space W, but now the Hg-action is $\rho_g(x) = \rho(h)$, where $x \in gHg^{-1}$. Since $H_g \leq K$, we obtain an induced represntation $Ind_{H_g}^G W_g$ from this.

Theorem. (12.4) (Mackey's restriction formula)
G finite, $H, K \leq G$ and W H-space. Then

$$Res_K^G Ind_H^G W = \oplus_{g \in S} Ind_{H_g}^K W_g$$

as K-modules.
We'll prove this later.

Corollary. (12.5, character version of (12.4))
If ψ is a character of a representation of H, then

$$es_K^G Ind_H^G \psi = \sum_{g \ni S} Ind_{H_g}^K \psi_g$$

where ψ_g is the character of H_g given as $\psi_g(x) = \psi(g^{-1}xg)$.

Corollary. (12.6, Mackey's irreducibility criterion)
Let $H \leq G$, W be a H-vector space. Then $V = Ind_H^G W$ is irreducible iff
(i) W is irreducible;
(ii) for each $g \in S \setminus H$, the two Hg- spaces Wg and $Res_{H_g}^H W$ have no irreducible consitutnets in common (they're 'disjoint' representations).

Proof. Let W afford character ψ. Recall W irreducible $\iff$ $\langle \psi, \psi \rangle = 1$. Take $K = H$ in (12.4), so $Hg = gHg^{-1} \cap H$. Then

$$\langle Ind_H^G \psi, Ind_H^G \psi \rangle_G = \langle \psi, Res_H^G Ind_H^G \psi \rangle_H$$

by (10.7), then by (12.5) is equal to

$$\begin{aligned} \sum_{g \in S} \langle \psi Ind_{H_g}^H \psi_g \rangle_H &= \sum_{g \in S} \langle Res_{H_g}^H \psi, \psi_g \rangle_{H_g} \\ &= \langle \psi, \psi \rangle_H + \sum_{g \in S, g \notin H} d_g \end{aligned}$$

where $d_g = \langle Res_{H_g}^H \psi, \psi_g \rangle$ $(g \neq 1)$. □

For $g = 1$ we have $H_g = H$, hence we get a sum of non-negative integers which is ≥ 1. So Ind_H^ψ is irreducible iff $\langle \psi, \psi \rangle = 1$ and all the other terms in the sum are 0. In other words, W is irreducible representation of H and $\forall g \notin H$, W and W_g are disjoint representations of $H \cap gHg^{-1}$.

Remark. Set S of representations was arbitrary, so could demand $g \in G \setminus H$ in (ii) but in fact suffices to check for $g \in S \setminus H$.

Corollary. (12.7)
If $H \triangleleft G$, assume ψ is an irreducible character of H. Then $Ind_H^G \psi$ is irreducible $\iff$ ψ is distinct from all its conjugates ψ_g for all $g \in G \setminus H$ ($\psi_g(h) = \psi(ghg^{-1})$).

Proof. Again take $K = H$, noting double cosets $\equiv$ left cosets. Also, $Hg = H$ $\forall g$ (as $H \triangleleft G$). Moreover, Wg is irreducible since W is irreducible. So by (12.6), $Ind_H^G W$ is irreducible precisely when $W \not\cong Wg$ $\forall g \in G \setminus H$. This is equivalent to $\psi \neq \psi g$. □

Remark. Again could check conditions on a set of representatives.

Proof. (of 12.4)
Write $V = Ind_H^G W$. Fix $g \in G$. Now V is a direct sum of $x \oplus W$ with x running

through representations of left cosets of H in G (10.11). $V = \oplus_{x \in J} x \otimes W$. Consider a particular coset $KgH = K \backslash G/H$. The terms

$$V(g) = \oplus_{x \text{ rep of H in } G, x \in KgH} x \otimes W$$

forms a subspace invariant under the action of K (it's a direct sum of an orbit of subspaces permuted by K). Now viewing V as a K-space (forget G-structure), $Res^G_K V = \oplus_{g \in S} V(g)$, so we need to show $V(g) = Ind^K_{H_g} W_g$ as K-spaces for each $g \in S$.

Now, $Stab_K(g \otimes W) = \{k \in K : kg \otimes W = g \otimes W\} = \{k \in K : g^{-1}kg \in Stab_G(1 \otimes W) = H\} = K \cap gHg^{-1}$ $(= Hg)$. This implies if $x = kgh, x' = k'gh'$, then $x \otimes W = x' \otimes W$ iff k, k' lie in same coset in K/Hg, hence $V(g)$ is direct sum $\oplus_{k \in K/Hg} k \otimes (g \otimes W)$. Therefore, as a representation of K, this subspace is

$$V(g) \cong Ind^K_{H_g}(g \otimes W)$$

□

But $g \otimes W \cong Wg$ as a representaiton of Hg ($w \to g \otimes W$ is an isomorphism). Putting everything together we are done.

13 Integrality in the group algebra

Definition. (13.1)
$a \in \mathbb{C}$ is an algebraic integer if: a is a root of a monic polynomial in $\mathbb{Z}[x]$. Equivalently, the subring of $\mathbb{C}$ generated by $\mathbb{Z}[a] = \{f(a) : f(x) \in \mathbb{Z}[x]\}$ is a finitely generated $\mathbb{Z}$-module.

Fact 1: The algebraic integers form a subring of $\mathbb{C}$ (see number fields);
Fact 2: If $a \in \mathbb{C}$ is both an algebraic integer and a rational number, then it's an integer (see number fields);
Fact 3: Any subring of $\mathbb{C}$ which is a finitely-generated $\mathbb{Z}$-module consists of algebraic integers.

Proposition. (13.2)
If χ is character of G and $g \in G$, then $\chi(g)$ is an algebraic integer.

Proof. $\chi(g)$ is a sum of nth roots of unity ($n = |g|$). Each root of unity is an algebraic integer, and any sum of algebraic integers is an algebraic integer by fact 1. □

Corollary. There are no entries in the chracter rables of any finite group which are rational but not integers, by Fact 2.

13.1 The centre of $\mathbb{C}G$

Recall from (2.4), the group algebra $\mathbb{C}G = \{\sum \alpha_g g : \alpha_g \in \mathbb{C}\}$ of finite group, the $\mathbb{C}$-space with basis G. Also a ring, hence a finite-dimensional $\mathbb{C}$-algebra.

List $\{1\} = \mathcal{C}_1, ..., \mathcal{C}_k$ the G-ccls. Define the class sums:

$$C_j = \sum_{g \in \mathcal{C}_j} g \in \mathbb{C}G$$

Claim, each $C_j \in Z(\mathbb{C}G)$, the centre of $\mathbb{C}G$ (Note: this is not the same as $\mathbb{C}(Z(G))$!).

Proposition. (13.3)
$C_1, ..., C_k$ is a basis of $Z(\mathbb{C}G)$. There exist non-negative integers a_{ijl} ($1 \leq i, j, l \leq k$) with

$$C_i C_j = \sum_l a_{ijl} C_l$$

These are called the class algebra constants for $Z(\mathbb{C}G)$.

Remember last time we had $C_j = \sum_{g \in \mathcal{C}_j} g$ $(= \sum_{k=1}^r x_k^{-1} g_j x_k)$. We claimed that $c_1, ..., c_k$ are basis for $Z(\mathbb{C}G)$. Let's now prove it.

Proof. Check that $gC_jg^{-1} = C_j\ \forall g \in G$. So $C_j \in Z(\mathbb{C}G)$. Clear that the C_j are linearly independent (because the $\mathcal{C}_J$ are pairwise disjoint). Now suppose $z \in Z(\mathbb{C}G)$, $z = \sum_{g\in G} \alpha_g g$. Then $\forall h \in G$ we have $\alpha_{h^{-1}gh} = \alpha_g$. So the function $g \to \alpha_g$ is constant on G-conjugacy classes. Wrigting $\alpha_g = \alpha_j$ for $g \in \mathcal{C}_j$, then $z = \sum_1^k \alpha_j C_j$. Finally, $Z(\mathbb{C}G)$ is a $\mathbb{C}$-algebra, so $C_iC_j = \sum_{l=1}^k \underbrace{a_{ijl}}_{\in\mathbb{C}} C_l$, as the C_l span. We claim $a_{ijl} \in \mathbb{Z}_{\geq 0}\ \forall i,j,l$: Fix $g_l \in \mathcal{C}_l$. Then

$$a_{ijl} = \text{number of}\{(x, y \in \mathcal{C}_i \times \mathcal{C}_j : xy = g_l\} \in \mathbb{Z}_{\geq 0}$$

□

Definition. (13.4)
Let $\rho : G \to GL(V)$ be an irreducible representation over $\mathbb{C}$, affording character χ. Extend by linearity to $\rho : A = \mathbb{C}G \to End_{\mathbb{C}}V$, an algebra homomorphism. Any homomorphism of algebras $A \to EndV$ is called a representation of A. A *central character* of A is a ring homomorphism $Z(A) \to \mathbb{C}$. Let $z \in Z(\mathbb{C}G)$. Then $\rho(z)$ commutes with all $|rho(g)$ $(g \in \mathbb{C}G)$, so by Schur's lemma, $\rho(z) = \lambda_z I$ for some $\lambda_z \in \mathbb{C}$. Now consider the algebra homomorphism $\omega_\chi = \omega : Z(\mathbb{C}G) \to \mathbb{C}$ by $z \to \lambda_z$. Now $\rho(C_i) = \omega(C_i)I$, so, taking traces,

$$\chi(1)\omega(C_i) = \sum_{g\in\mathcal{C}_i} \chi(g) = |\mathcal{C}_i|\chi(g_i)$$

where g_i is a representation of $\mathcal{C}_i$. So $\omega(C_i) = \frac{\chi(g_i)}{\chi(1)}|\mathcal{C}_i|$.

Lemma. (13.5)
The values $\omega(C_i) = \frac{\chi(g)}{\chi(1)}|\mathcal{C}_i|$ are algebraic integers.

Proof. Since ω is an algebra homomorphism and using (13.3),

$$\omega(C_i)\omega(C_j) = \sum_{l=1}^k a_{ijl}\omega(C_l)$$

where $a_{ijl} \in \mathbb{Z}_{\geq 0}$. Thus the span $\{\omega(C_l) : 1 \leq l \leq k\}$ is a subring of $\mathbb{C}$ and is a finitely-generated abelian group, so by Fact 3, consists of algebraic integers.
[A bit of explanations:

$$\omega(C_i)\omega(C_j) = \sum a_{ijl}\omega(C_l)$$
$$\omega(C_i)\begin{pmatrix}\omega(C_1)\\ \dots\\ \omega(C_k)\end{pmatrix} = (a_{ijk})\begin{pmatrix}\omega(C_1)\\ \dots\\ \omega(C_k)\end{pmatrix}$$

$\omega(C_i)$ is eigenvalue of the integer matrix (a_{ijl}) so an algebraic integer by definition.] □

Exercise (Burnside, 1911):
Show that a_{ijl} can be obtained from the charcater table. In fact, $\forall i,j,l$,

$$a_{ijl} = \frac{|G|}{|C_G(g_i)||C_G(g_j)|}\sum_{s=1}^k \frac{\chi_s(g_i)\chi_s(g_j)\chi_s(g_l^{-1})}{\chi_s(1)}$$

for $g_i \in \mathcal{C}_i$, $1 \le i \le l$.
(proof uses column orthogonality, JL 30.4).

Theorem. (13.6)
The degree of any irreducible charcaters of G divides $|G|$.

Proof. Given irreducible charcater χ, apply orthogonality,

$$\begin{aligned}\frac{|G|}{\chi(1)} &= \frac{1}{\chi(1)} \sum_{g \in G} \chi(g)\chi(g^{-1}) \\ &= \frac{1}{\chi(1)} \sum_{i=1}^{k} |\mathcal{C}_i| \chi(g_i)\chi(g_i^{-1}) \\ &= \sum_{i=1}^{k} \frac{|\mathcal{C}_i \chi(g_i)}{\chi(1)} \chi(g_i^{-1})\end{aligned}$$

where in the last summand, the first fraction is an algebraic integer by (13.5), and $\chi(g_i^{-1})$ is sum of roots of unity so an algebraic integer. LHS is clearly also rational, so it's an integer. □

Example. (13.7)
(a) If G is a p-group, then $\chi(1)$ is a p-power (χ irreducible). In particualr, if $|G| = p^2$, then $\chi(1) = 1$ (since we already have a trivial character – the idea is actually similar to the proof in Groups 1A), hence G abelian.
(b) If $G = S_n$ then every prime dividing the degree of an irreducible charcater of G also divides $n!$.

Theorem. (13.8, Burnside, 1904)
If χ is irreducible, then $\chi(1) | \frac{|G|}{|Z|}$.
The proof is left as an exercise. As a hint, it uses tensor products.

14 Burnside's theorem

Theorem. (14.1)
Let p, q be primes, let $|G| = p^a q^b$, where $a, b \in \mathbb{Z}_{\geq 0}$, with $a + b \geq 2$. Then G is not simple.

Proof. The theorem gollows from 2 lemmas. We will prove this on Saturday. □

Remark. (1) In fact, even more is true: G is soluble.
(2) Result is best possible, in the sense that $|A_5| = 60 = 2^2 \cdot 3 \cdot 5$ has 3 prime factors, and is simple (actually there are 8 non-soluble groups of order $p^a q^b r^c$ for p, q, r primes).
(3) If either a or b is 0 then G is a p group, so is nilpotent, so soluble.
(4) In 1963, Feit and Thompson proved that every group of odd order was soluble.