Graph Theory

December 8, 2017

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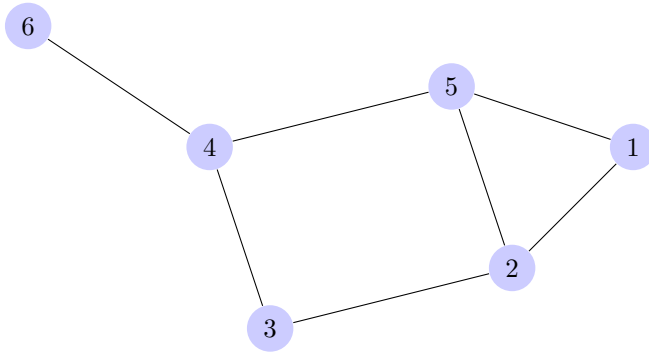
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0 Introduction

Some not very useful examples.

1 Ramsey Theory

Definition. A *graph* is an ordered pair $(V, E) = G$ where V is a finite set and E is a set of unordered pairs of distinct elements of V. We call elements of V *vertices* of G and elements of E *edges*.

We often write $v \in G$ to mean $v \in V$ and similarly for edges as well. Often we denote $\{u, v\} \in E$ by uv.

Example. $G = (\{1, 2, 3, 4, 5, 6, 7\}, \{12, 23, 13, 14, 67\})$ is a graph. We can represent a graph by a drawing: take a point for each vertex, join two vertices if they are in an edge.
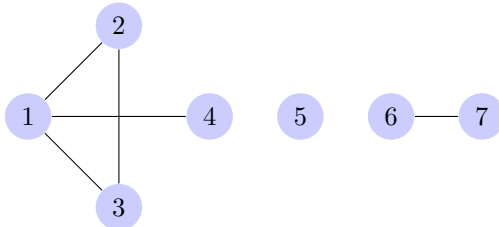
Definition. Let $G = (V, E)$ and $G' = (V', E')$ be graphs. An *isomorphism* from G to G' is a bijection $\varphi : V \to V'$ such that for all $u, v \in V$, we have $\phi(u)\phi(v) \in E' \iff uv \in E$.

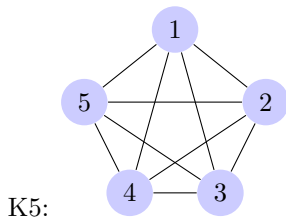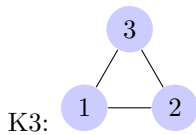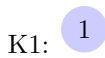If such an isomorphism exists, we say G is isomorphic to G'.

Suppose also $H = (W, F)$ is a graph. We say H is a *subgraph* of G and write $H \subset G$ if $W \subset V$ and $F \subset E$.

Often we say H is a subgraph of G to mean H is isomorphic to a subgraph of G.

Example. (Complete graph)
The *complete graph of order n*, K_n, has n vertices with every pair forming an edge.

K1:

K3:

K5:

Note that if $m \leq n$ then $K_m \subset K_n$.

Recall *Schur's Theorem*: Let k be a positive integer. Then there is a positive integer n s.t. if the set $[n] = \{1, 2, ..., n\}$ is coloured with k colours, we can find a, b, c with $a + b = c$ and a, b, c the same colour. An idea to prove this in graph theory is to consider vertices as numbers, edges as difference between two vertices, and we try to colour the edges.

For example, suppose the edges of complete graph K_6 are coloured blue/yellow. Then we can find a monochromatic triangle. Note that Schur's Theorem for $k = 2$ follows immediately. In this graph form of the problem statement, statement for more colours is amenable to an induction proof.

Proposition. Let k be a positive integer. Then there is a positive integer n s.t. whenever the edges of K_n are coloured with k colours, we can find a monochromatic triangle.

Proof. We use induction on k. $k = 1$ is trivial with $n = 3$. For $k > 1$, by induction hypothesis we can find a m s.t. for any K_m coloured with $(k - 1)$ colours, there exist a monochromatic triangle.
Let $n = k(m-1)+2$. Now k-colour the edges of K_n. Pick vertex v. The number of edges containing v is $n - 1 = k(m - 1) + 1$, so some m of them are the same colour, wlog blue. Now let H be a K_m joined to v by blue edges. If H contains a blue edge then we have a blue triangle. Otherwise H is only coloured with at most $k - 1$ colours, and from the induction hypothesis we know that there is a monochromatic triangle in that K_m. $\qquad\square$

Remark. We can also try to get an explicit upper bound from what we proved above. Let $f(k)$ be the smallest n that works with k colours. Clearly $f(1) = 3$. The proof says for $k > 1$, $f(k) \leq k(f(k-1)-1)+2 \leq kf(k-1)$. So by induction we have $f(k) \leq 3k!$ for all k.

Remark. Schur's Theorem follows immediately if we consider the graph approach of the problem.

Another problem that arises naturally is that, can we find larger monochromatic subgraphs, e.g. a K_4? We can think about the two colours case first for simplicity – the answer is yes, but it requires some intermediate step.

Example. Let K_{10} have edges coloured blue/yellow. Then there is a blue triangle or yellow K_4.

Proof. Take a vertex v. It's degree 9, so either v has 4 blue edges, in which case there is a blue triangle or yellow K_4; otherwise v has 6 yellow edges, in which case there is a blue or yellow triangle within it, together with v we have either a blue triangle or yellow K_4. $\qquad\square$

Proposition. Let K_{20} be coloured blue/yellow. Then there exists a monochromatic K_4.

Proof. Pick a vertex v. It's degree 19 so 10 of them are the same colour, wlog blue. Then by the previous example we get the desired result immediately. $\square$

Now we are interested in a more general problem.

Definition. Let $s, t \geq 2$. The *Ramsey number* $R(s, t)$ is the least n s.t. whenever K_n has edges coloured blue or yellow, ther must be a blue K_s or a yellow K_t, provided if such an n exists. Also we write $R(s) = R(s, s)$ for simplicity.

Earlier we've seen $R(3) = 6$, $R(3, 4) \leq 10$, $R(4) \leq 20$.

Theorem. (Ramsey's theorem)
$R(s, t)$ exists for all $s, t \geq 2$. Moreover, is $s, t > 2$, then $R(s, t) \leq R(s - 1, t) + R(s, t - 1)$.

Proof. We'll prove by induction on $s + t$.

First consider the case $s = 2$: Obviously $R(2, t) = t$. The case $t = 2$ is symmetric.

Now if $s, t > 2$, Let $a = R(s - 1, t)$ and $b = R(s, t - 1)$ (and we know by induction hypothesis that these exist).

Now let $n = a + b$, and we colour edges of K_n by blue and yellow. Pick $v \in K_n$, it has either a blue or b yellow edges. Then we can just apply the induction hypothesis. $\square$

And we have an immediate corollary:

Corollary. For all $s, t \geq 2$, $R(s, t) \leq 2^{s+t}$. So $R(s) \leq 4^s$.

Proof. We induct on $s + t$. The case $s = 2$ or $t = 2$ is obvious. For $s, t > 2$ we have
$$
\begin{aligned}
R(s, t) &\leq R(s - 1, t) + R(s, t - 1) \\
&\leq 2^{s-1+t} + 2^{s+t-1} \\
&= 2^{s+t}.
\end{aligned}
$$
$\square$

We can prove that $R(s) = O\left(\frac{4^s}{\sqrt{s}}\right)$ by the same proof but more carefully – this was first proved in 1935. After 50 years we proved that $R(s) = O\left(\frac{4^s}{s}\right)$ by being very clever and working hard. After another 20 years (2009), by being even more careful and working harder, we can prove $R(s) = O\left(\frac{4^s}{s^k}\right)$ for any k. It is still unknown that if the base in the expression can be reduced by any tiny amount.

Theorem. (Multi-colour Ramsey Theorem)
Let $k \geq 1$ and $s \geq 2$. Then there exists some n s.t. whenever the edges of K_n coloured with k colours, we can find a monochromatic K_s.

Proof. Use induction on k. For $k = 1$ we just take $n = s$.

Now let $k > 1$. By induction hypothesis, we can find M s.t. K_m coloured with $(k-1)$ colours contain a monochromatic K_s. Now let $n = R(s, m)$ and colour K_n with k colours, including blue but not yellow. Now we re-colour by turning all the non-blue edges yellow. Now by definition of n we have either a blue K_s, in which case we're done, or we have a yellow K_m. But that means we have a K_m coloured with $k-1$ colours, which by induction hypothesis we can also find a monochromatic K_s. $\square$

Remark. We write $R_k(s)$ for the smallest n that works. In particular, $R_1(s) = s$, $R_2(s) = R(s)$. Proof of the above theorem will give $R_k(s) \leq 4^{s^{k-1}}$ (see example sheet 1).

Remark. What we really mean by colouring is that we have a function $c : E \to [k] = \{1, 2, ..., k\}$. 'Blue' and 'yellow' are just arbitrary names of the colours.

We say $H = (W, F)$ is monochromatic if $c|_F$ is a constant.

With any number of colours, we can find arbitrarily large monochromatic subgraphs. What about infinite ones?

Definition. An infinite graph is an ordered pair $G = (V, E)$ where V is an infinite set and E is a set of unordered pairs of distinct elements of V. Note that in our terminology, an infinite graph is *not* a graph.

We should also let a possibly infinite graph mean a graph or an infinite graph.

The *infinite complete graph*, K_∞, will mean the infinite graph with a *countably* infinite vertex set and every pair of vertices forming an edge.

Where it makes sense to do so, we carry terminology over from graphs to infinite graphs.

Suppose edges of K_∞ are coloured blue or yellow. What can we find? By Ramsey, we can find a monochromatic K_s for every finite $s = 1, 2, 3, ...$ This doesn't imply we find an infinite monochromatic subgraph.

Theorem. (5, Infinite Ramsey Theorem)
Let $k \geq 1$. Whenever the edges of K_∞ are k-coloured, we have a monochromatic K_∞ as a subgraph.

Proof. Take $v_1 \in K_\infty$. The vertex v_1 is infinitely many edges, so infinitely of them are the same colour. Let $A_1 \subset V$ be the infinite subset of vertices of K_∞ s.t. for all $u \in A_1$, $v_1 u$ has colour c_1.

Now pick $v_2 \in A_1$. Similarly, we can find an infinite $A_2 \subset A_1$ s.t. all edges $v_2, u(u \in A_2)$ have colour (WLOG) c_2. Keep going and we get infinite sequences $v_1, v_2, v_3, ...$ of vertices, $c_1, c_2, c_3, ...$ of colours and $A_1 \supset A_2 \supset A_3 \supset ...$ of infinite sets s.t.

- for $i \geq 2$, $v_i \in A_{i-1}$;
- for $i \geq 1$, for all $u \in A_i$, $v_i u$ is an edge of colour c_i.

In particular, if $i < j$ then $v_i v_j$ has colour c_i. However there are only finitely many colours. So infinitely many of the c_i are the same, say we have $c_{i_1} = c_{i_2} = ...$ Then $v_{i_1}, v_{i_2}, ...$ form a monochromatic K_∞. □

An application of the above theorem is

Corollary. (6, Bolzano-Weierstrass Theorem)
Any bounded sequence has a convergent subsequence (apparently, in a sequentially compact space).

Proof. Any bounded monotonic sequence converges, so it's enough to show that any real sequence $(x_n)_{n \geq 1}$ has a monotonic subsequence. Let G be a K_∞ with vertex set $\{1, 2, 3, ...\}$. Colour ij $(i < j)$ blue if $x_i < x_j$, and yellow otherwise. By theorem 5 there is a monochromatic subgraph isomorphic to K_∞, so there is a monotonic subsequence. □

2 Basic Terminology

Let $G = (V, E)$ be a graph. Sometimes we write $V(G) = V$ and $E(G) = E$.

The *order* of G is $|G| = |V(G)|$. We denote the number of edges of G by $e(G) = |E(G)|$, sometimes this is called the 'size' of G.

Now let $v \in G$. The *neighbourhood* of v is the set $\Gamma(v) = \{w \in G : vw \in E(G)\}$.

If $w \in \Gamma(v)$ then w is a *neighbour* of v, or w is *adjacent* to v, written as $w \sim v$.

The *degree* of v is $d(v) = |\Gamma(v)|$, i.e. the number of vertices adjacent to v. From this it should be obvious what the maximum degree, minimum degree, and average degree mean. We denote the maximum and minimum degree by $\Delta(G)$, $\delta(G)$ respectively.

Clearly $\delta(G) \leq$ average degree $\leq \Delta(G)$. If every vertex in G has the same degree, we say G is *regular*. If this degree is r, we say G is $r-$regular.

Example. Each edge contributes 2 to the sum of degrees, so we know $2e(G)$ is equal to $\sum_{v \in G} d(v)$.

Let G be a graph. A *path* in G is a finite sequence $v_0, ..., v_l$ of distinct vertices of G with $v_{i-1} \sim v$ for $1 \leq i \leq l$. We say this path has length l and goes from v_0 to v_l. We write $v \rightarrow w$ to mean there is a path from v to w. The relation $\rightarrow$ is an equivalence relation. The equivalence classes of $\rightarrow$ are called the *components* of G. If G has only one component, we say G is *connected*.

A *cycle* is a sequence $v_0, ..., v_l$ of vertices of G with $v_0, ..., v_{l-1}$ distinct, $v_l = v_0$, $v_{i-1} \sim v_i$ for $1 \leq i \leq l$, and $l \geq 3$. We say the length of the cycle is l.
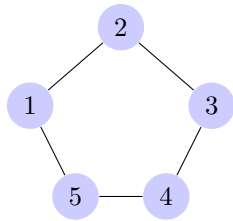
A graph with no cycles is called a *forest*. A *tree* is a connected forest. Each component of a forest is a tree.

Suppose G, H are graphs with $V(G) \cap V(H) = \phi$. The *disjoint union* of G, H is the graph $G \cup H$, with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$.

Often we write $G \cup H$ even if $V(G) \cap V(H) \neq \phi$. This means that we take graphs G', H' that are isomorphic to G and H, s.t. $V(G') \cap V(H') = \phi$, then take $G' \cup H'$.

Example. (i) $K_3 \cup K_5$:
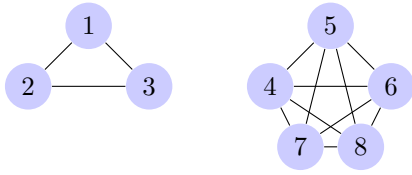
(ii) Any graph is the disjoint union of its components. For example, a forest is a disjoint union of trees.

Let $G = (V, E)$ be a graph and let $W \subset V$. The *induced subgraph on W* is the graph $G[W]$ with $V(G[W]) = W$ and, for $x, y \in W$, $xy \in E(G[W]) \iff xy \in G$.

Let $G = (V, E)$ be a graph. The *complement* of G is the graph $\bar{G}$ with $V(\bar{G}) = V$ and for distinct $x, y \in V$, $xy \in E(\bar{G}) \iff xy \notin E$.

Example. • Suppose edges of Kn are coloured blue/yellow. We can just think about the 'blue subgraph' with n vertices and blue edges only. The complement is then the yellow subgraph.
• For example, consider graph G
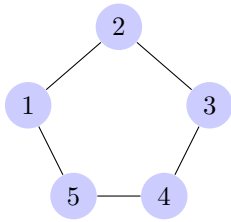
Then we have $\bar{G}$ is
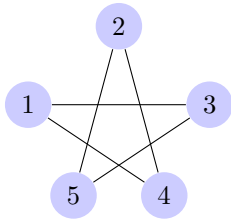
and by relabelling the vertices we can see that G is actually isomorphic to its complement.

Reminder of O notation: Let $f, g : \mathbb{N} \to (0, \infty)$. We say $f = O(g)$ if $f < Ag$ for some constant, say $f = \Omega(g)$ if $g = O(f)$, say $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$. Also, $f = o(g)$ means $f/g \to 0$ (as $n \to \infty$), $f = \omega(g)$ means $f/g \to \infty$, and $f \sim g$ means $f/g \to 1$.

3 Extremal Graph Theory

Problem 1 (from chapter 1): How large must n be so that if edges of K_n coloured blue/yellow then we always get a monochromatic K_s? We can actually let G be the blue subgraph of K_n, then K_n has a blue K_s is equivalent to $K_s \subset G$, and K_n has a yellow K_s is equivalent to G has an induced $\bar{K}_s$ subgraph. Then we can rephrase the problem as: How large must n be to force every subgraph of order n to have K_s or $\bar{K}_s$ as an induced subgraph?

Typical example of an extremal problem: how large must some parameter of G be to force G to have a certain property? Alternatively, how big can the parameter be with G not having the property?

3.1 The Forbidden Subgraph Problem

Fix a graph H with at least one edge. Let $n \geq |H|$. Clearly $H \subset K_n$ but $H \not\subset \bar{K}_n$. How many edges must a graph G of order n have to force $H \subset G$? Alternatively, if $|G| = n$ and $H \subseteq G$, how large can $e(g)$ be?

Definition. Define

$$ex(n, H) = \max\{e(G) : |G| = n, H \not\subset G\}$$

Problem: can we determine $ex(n, H)$?

3.1.1 Triangles

We want G with $|G| = n$, $e(G)$ large, and $\triangle \not\subset G$. The idea is to have a *bipartite graph*, where we can partition the graph G into two parts X and Y such that there is no edge within X or Y themselves, i.e. with all the edges going from X to Y. We define this more properly:

Definition. (Bipartite graph)
A graph G is *bipartite* (with bipartition (X, Y)) if $V(G)$ can be partitioned as $X \cup Y$ in such a way that if $e \in E(G)$, then $e = xy$ for some $x \in X, y \in Y$.

Bipartite graphs have no triangles (and indeed no cycles of odd length). In fact, the converse is true.

Theorem. (7)
A graph is bipartite iff it contains no odd cycles. The proof is an exercise for now.

However, which bipartite graph is the best? Clearly we want to include all possible edges from X to Y.

Definition. Let $s, t \geq 1$. The complete bipartite graph $K_{s,t}$ has bipartition (X, Y) with $|X| = s$, $|Y| = t$ and $xy \in E(K_{s,t}) \ \forall x \in X, y \in Y$.

We have $|K_{s,t}| = s + t$ and $e(K_{s,t}) = st$. So we just have to maximize $s(n-s)$. Now we'll prove that no other graphs can beat bipartite graphs.

Theorem. (Mantel's theorem)
Let $n \geq 3$. Suppose $|G| = n$, $e(G) \geq [\frac{n^2}{4}]$ and G does not contain a triangle. Then $G \cong K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

Proof. By induction. For $n = 3$ it's trivial.
For $n > 3$, let $|G| = n$, $e(G) \geq \lceil \frac{n^2}{4} \rceil$, and G does not contain a triangle. First, remove edges from G if necessary to get H with $|H| = n$, $e(H) = \lfloor \frac{n^2}{4} \rfloor$. Clearly H does not contain a triangle. Let $v \in H$ with $d(v) = \delta(H)$ and let $K = H - v$, i.e. H with vertex v and all edges including v removed. (recall that $\delta(H)$ is the minimum degree). Now $|K| = $ n-1, K does not contain a triangle, and $e(K) = \lfloor \frac{n^2}{4} \rfloor - \delta(H)$.

Suppose n is even. Then $\delta(H) <$average degree of $H = 2e(H)/|H| = n^2/2/n = n2$, hence
$$e(K) \geq \frac{n^2}{4} - \frac{n}{2} = \frac{(n-1)^2}{4} = \lfloor \frac{(n-1)^2}{4} \rfloor$$
Similarly if n is odd we also get $e(K) \geq \lfloor \frac{(n-1)^2}{4} \rfloor$. Hence by induction hypothesis, $K \cong K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$.

Also, $d(v) = e(H) - e(K)$. If n is even then $d(v) = n/2$. H is formed by adding a vertex v to $K \cong K_{\frac{n}{2}, \frac{n-2}{2}}$, and joining v to $n/2$ vertices of K without creating a triangle. If K has bipartition (X, Y), v cannot be joined both to a vertex in X and a vertex in Y. So v must be joined to all vertices in the larger of X, Y. Thus $H \cong K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$, and similar if n is odd. We recover G by adding edges to H without making a triangle. But any new edge creates a triangle, so $G \cong H$. □

Hence if $|G| = n, e(G) > \lfloor \frac{n^2}{4} \rfloor$, then G contains a triangle. Therefore $ex(n, \triangle) = \lfloor \frac{n^2}{4} \rfloor$.

3.1.2 Complete graphs

Last time we considered K_3, the triangle. What about K_4? We can try something like a 'tripartite graph'. Formally,

Definition. A graph G is r-partite if we can partition $V(G) = X_1 \cup ... \cup X_r$ in such a way that if $xy \in E(G)$, then $x \in X_i, y \in X_j$ for some i, j with $i \neq j$. We say G is complete r-partite graph if whenever $x \in X_i, y \in X_j$ with $i \neq j$, we have $xy \in E(G)$.
The Turan graph $T_r(n)$ is the complete r-paritte graph with n vertices and vertex-classes as equal as possible. We write $t_r(n) = e(T_r(n))$.

Properties of the Turan graph:

- $K_{r+1} \not\subset T_r(n)$. If we add an edge to $T_r(n)$, we create a K_{r+1}.

3.1.3 Bipartite graphs

Example. What is $ex(n, C_4)$? Note that $C_4 \cong K_{2,2}$.
C_4 is made from two copies of P_2. Suppose G is a graph, $|G| = n$, $e(G) = m$, $C_4 \not\subset G$. How many P_2 subgraphs does G have? Given $v \in G$, v is middle vertex of $\binom{d(v)}{2}$ P_2 s, so total number of P_2 is $\binom{\sum_{v \in G} d(v)}{2}$. Instead, observe if $\{x, y\} \subset V(G)$ with $x \neq y$ then at most one P_2 with x, y as end-vertices (since $C_4 \not\subseteq G$). So number of P_2 is at most $\binom{n}{2}$. Hence $\sum_{v \in G} \binom{d(r)}{2} \leq \binom{n}{2}$.

Recall Jensen's inequality: for a convex function f, we have

$$\frac{1}{n} \sum_{i=1}^{n} f(x_i) \geq f\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right)$$

Applying Jensen's inequality to the previous upper bound we have, we get

$$\binom{n}{2} \geq \sum_{v \in G} \binom{d(v)}{2} \geq n \binom{\frac{1}{n} \sum_{v \in G} d(v)}{2} = n \binom{2m/n}{2}$$

(recall $m = e(G)$). Hence $n(n-1) \geq n \cdot (2m/n) \cdot (2m/n - 1)$, so $n^2(n-1) \geq 2m(2m - n)$, i.e.

$$4m^2 - 2mn - n^2(n-1) \leq 0$$

This is a quadratic equation in m, the root are

$$m = \frac{2n \pm \sqrt{4n^2 + 16n^2(n-1)}}{8}$$
$$= \frac{n}{4}\left(1 \pm \sqrt{1 + 4(n-1)}\right)$$
$$= \frac{n}{4}\left(1 \pm \sqrt{4n-3}\right)$$

Hence $m \leq \frac{n}{4}(1 + \sqrt{4n-3})$, i.e. $ex(n, C_4) \leq \frac{n}{4}(1 + \sqrt{4n-3})$, which means $ex(n, C_4) = O(n^{3/2})$.

The same approach turns out to give a bound on $ex(n, K_{t,t})$ in general. For example, if we consider $K_{3,3}$, we consider a vertex as the end-vertex for 3 edges. (not very obvious)

Definition. A $t-$fan in a graph G is an ordered pair (v, W), where $v \in V(G)$, $W \subset V(G)$, $|W| = t$ and $\forall w \in W$, $v \sim w$ (there is an edge between them).

For example, a 2-fan is just a P_2 subgraph.

Theorem. (11)
Let $t \geq 2$. Then $ex(n, K_{t,t}) = O(n^{2-1/t})$.

Proof. Let $|G| = n$, $e(G) = m = ex(n, K_{t,t}$ and $K_{t,t} \not\subseteq G$. How many t-fans are there in G? Each $v \in G$ is in $\binom{d(v)}{t}$ t-fans. Thus total number if $\sum_{v \in G} \binom{d(v)}{t}$.

Given $W \subset V(G)$ with $|W| = t$, W is in at most $t - 1$ t-fans (as $K_{t,t} \not\subset G$). So total number of t-fans is at most $\binom{n}{t}(t-1)$. Hence

$$\binom{n}{t}(t-1) \geq \sum_{v \in G} \binom{d(v)}{t}$$

$$\geq n\binom{\frac{1}{n}\sum_{v \in G} d(v)}{t}$$

$$= n\binom{2m/n}{t}$$

Note that $\binom{n}{t} \geq n^t/t!$. So

$$\frac{n^t}{t!}(t-1) \geq \frac{n}{t!}\left(\frac{2m}{n} - t\right)^t$$

get rid of the $t!$,

$$n^t(t-1) \geq n\left(\frac{2m}{n} - t\right)^t$$

$\square$

If n is sufficiently large (as we may assume), then $m \geq nt$ (check by construction), so $\frac{m}{n} \geq t$ and so $\frac{2m}{n} - t \geq \frac{m}{n}$. Thus $n^t(t-1) \geq n(m/n)^t$. Rearrange this we get $m^t \leq n^{2t-1}(t-1)$. So $m \leq (t-1)^{1/t}n^{2-1/t}$. t is just a constant, so we get $m = O(n^{2-1/t})$.

So we've got an upper bound. We'll discuss about lower bounds later.

Closely related is the *problem of Zarankiewicz*: let $z(n,t)$ be the largest possible number of edges in a bipartite graph G with n vertices in each class and $K_{t,t} \not\subset G$ (and obviously the problem is to find $z(n,t)$).

Theorem. Let $t \geq 2$. Then $z(n,t) = O(n^{2-1/t})$.

Proof. This proof is similar with the previous, so we will not go into the details here.
Let G be bipartite with classes X, Y with $|X| = |Y| = n$, $e(G) = m = z(n,t)$ and $K_{t,t} \not\subset G$. We count t-fans with vertex in X and set in Y. Similarly,

$$\binom{n}{t}(t-1) \geq \sum_{v \in X} \binom{d(v)}{t}$$

Now $\sum_{v \in X} d(v) = m$. So similar calculation to theorem 11 gives $m = O(n^{2-1/t})$.
$\square$

3.1.4 General graphs

What is $ex(n, H)$ for general H? The exact answer is apparently too hard. What about some asymptotic estimations?

Usually $ex(n, H) \to \infty$ as $n \to \infty$. How about proportion of edges? E.g. could we find $\lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}}$? Does it even exist?

Proposition. (13)
Let H be a graph with at least 1 edge, and for $n \geq |H|$, let $x_n = \frac{ex(n;H)}{\binom{n}{2}}$. Then (x_n) converges.

Proof. Let $n \geq |H|$, let $|G| = n$, $e(G) = ex(n, H) = x_n \binom{n}{2}$ and $H \not\subset G$. For any $v \in G$, $|G - v| = n - 1$, and $H \not\subset G - v$, so $e(G - v) \leq ex(n-1, H) = x_{n-1}\binom{n-1}{2}$. Each edge $xy \in E(G)$ in $G - v$ for all $v \neq x, y$. Hence

$$(n-2)x_n\binom{n}{2} = (n-2)e(G) = \sum_{v \in G} e(G-v) \leq nx_{n-1}\binom{n-1}{2}$$

So x_n is decreasing. So it converges. $\square$

We write $ex(H) = \lim_{n\to\infty} \frac{ex(n;H)}{\binom{n}{2}}$, which exists because of the proposition above.

For Turan graph we have $ex(K_{r+1}) = 1 - \frac{1}{r}$.

From example sheet T11, we have $ex(n, K_{t,t}) = O(n^{2-1/t}) = o(n^2)$ So $ex(K_{t,t}) = 0$. Hence $ex(H) = 0$ for all bipartite graph H.

Can we determine $ex(H)$ in general?

The key fact is that, Turan says if we have proportion $1 - 1/r$ of edges, it's enough to guarantee a K_{r+1}. Increasing this proportion by a tiny amount gives *much* more.

Definition. Write $K_r(t)$ for complete r-partite graph with t vertices in each class. (So $K_r(t) = T_r(rt)$).

Theorem. (14, Erdös-Stone Theorem)
Let $r, t \geq 1$ be integers, and let $\varepsilon > 0$ be real. Then there exists n_0 s.t. for all $n \geq n_0$,

$$|G| = n, e(G) \geq \left(1 - \frac{1}{r} + \varepsilon\right)\binom{n}{2} \implies K_{r+1}(t) \subset G$$

This is enough to determine $ex(H)$ for all H.

Definition. If H is a graph, the *chromatic number* of H, denoted $\chi(H)$ is the least r such that H is r-partite. For exampl,e $\chi(K_r(t)) = r$.

Corollary. (15)
Let H be a graph with at least one edge. Then $ex(H) = 1 - \frac{1}{\chi(H)-1}$.

Proof. Let $r = \chi(H) - 1$. Then H is $(r+1)$-partite, so we can find t such that $H \subset K_{r+1}(t)$. (for example, $t = |H|$ suffices) Let $\varepsilon > 0$. Take n_0 as in theorem

14. Then for all $n \geq n_0$,

$$|G| = n, e(G) \geq (1 - \frac{1}{r} + \varepsilon)(\binom{n}{2})) \implies K_{r+1}(t) \subset G$$

$$\implies H \subset G$$

So for all $n \geq n_0$, $ex(n, H) < (1 - \frac{1}{r} + \varepsilon)\binom{n}{2}$. Hence $ex(H) = \lim_{n \to \infty} \frac{ex(n;H)}{\binom{n}{2}} \leq 1 - \frac{1}{r} + \varepsilon$. So $ex(H) \leq 1 - \frac{1}{r}$.

On the other hand, for all n, $H \not\subset T_r(n)$ as H not r-partite. So $ex(n; H) \geq t_r(n)$. As $n \to \infty$, $\frac{t_r(n)}{\binom{n}{2}} \to 1 - \frac{1}{r}$. So $ex(H) \geq 1 - \frac{1}{r}$. $\qquad \square$

So if H is not bipartiet, $ex(H) = \frac{1}{1 - \chi(H) - 1} \neq 0$, so $ex(n, H) \sim (1 - \frac{1}{\chi(H) - 1}\binom{n}{2})$.

So essentially we solved the forbidden subgraph problem for non-bipartite graph H.

For H bipartite, we only know $ex(n, H) = o(n^2)$. So it doesn't tell us if $ex(n, H)$ grows like a certain power of n.

We do, however, have upper bounds from T11: $ex(n, K_{t,t}) = O(n^{2-1/t})$.

Another application: we could define the *density* of a graph G to be $D(G) = e(G)/(\binom{|G|}{2})$. So $D(G) \in [0, 1]$. For example, $D(T_r(n)) \to 1 - \frac{1}{r}$ as $n \to \infty$.

If $\alpha > ex(H)$, then if $|G|$ sufficiently large and $D(G) = \alpha$, we have $H \subset G$.

We can also define something similar for infinite graphs.

Definition. The *upper density* of an infinite graph G is

$$ud(G) = \lim_{n \to \infty} \sup\{D(H) : H \subset G, |H| = n\}$$

again, $ud(G) \in [0, 1]$. If $\alpha < ud(G)$, then for sufficiently large n, G has subgraphs of order n and density $\geq a$. If $\alpha > ud(G)$, then for large n, G has no subgraphs of order n and density $\geq \alpha$.

A priori, it seems that $ud(G)$ could talke any value in $[0, 1]$. But a very surprising result is that

Corollary. (16)
For any infinite graph G,

$$ud(G) = \{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ...\}$$

are the only possible values.

Proof. We will show that for $r = 1, 2, 3, ...$, $ud(G) > 1 - \frac{1}{r}$ gives $ud(G) \geq 1 - \frac{1}{r+1}$.

Suppose $ud(G) > 1 - \frac{1}{r}$. Pick α s.t. $ud(G) > \alpha > 1 - \frac{1}{r}$. Fix n. For sufficiently large N, G has subgraphs of order N and density $\geq \alpha > 1 - \frac{1}{r} = ex(T_{r+1}(n))$. So $T_{r+1}(n) \subset G$. But $d(T_{r+1}(n)) \to 1 - \frac{1}{r+1}$ as $n \to \infty$.

We can do this for every n. So $ud(G) \geq 1 - \frac{1}{r+1}$. $\qquad\qquad\qquad\qquad\square$

3.1.5 Proof of Erdös-Stone Theorem

Theorem 14 (E-S) says if a graph G has enough edges then $K_{r+1}(t) \subset G$. Condition on $e(G) \leftrightarrow$ condition on average degree.

However, given d and $\delta > 0$, it tutrns out that any large graph of average degree $\geq d$ has a large subgraph of min degree $\geq d - \delta$. If G has average degree d and $|G| = n$ large, we can find $H \subseteq G$ with $|H| = \Omega(n)$ and $\delta(H) \geq d - \delta$ (not proved here. Delete vertices of min degree and it works.)

So we have an equivalent version of Erdös-Stone with minimum degree condition:

Theorem. (14a)
Let $r, t \geq 1$ be integers and $\varepsilon > 0$ be real. Then $\exists n_0 \forall n \geq n_0$, $|G| = n$, $\delta(G) \geq (1 - \frac{1}{r} + \varepsilon)n \implies K_{r+1}(t) \subset G$.

Proof. We do induction on r. Fix T such that $T > (\frac{2}{r\varepsilon})^t(t - 1)$. then choose n_0 s.t. for all $n \geq n_0$, $|G| = n$, $\delta(G) \geq (1 - \frac{1}{r} + \varepsilon)n \implies K_r(T) \subset G$ (How? when $r = 1$, $K_1(T) \subset G \iff |G| \geq T$, and when $r > 1$, $1 - \frac{1}{r} > 1 - \frac{1}{r-1}$, so follows from induction hypothesis)

Suppose the result is not true. Then we can find arbitrarily large n and graphs G with $|G| = n$, $\delta(G) \geq (1 - \frac{1}{r} + \varepsilon)n$, $K_{r+1}(t) \not\subset G$. Pick such an n, G with $n \geq n_0$ and also $n \geq \frac{2t}{r\varepsilon}$. Then we can find $K_r(T) \subset G$, say with vertex classes $X_1, ..., X_r$.

Let $A = \{(w, v_1, ..., v_r) : |w| = t, w \subset V(G) \forall i v_i \in X_i, \forall i, \forall w \in W, v_i \sim w\}$. What can we say about $|A|$?

First, given $v_1 \in X_1, ..., v_r \in X_r$, then we can check from minimum degree condition that $|\Gamma(v_1) \cap ... \cap \Gamma(v_r)| \geq r\varepsilon n$. So at least $\binom{r\varepsilon n}{t}$ choices for W. Hence $|A| \geq T^r \binom{r\varepsilon n}{t}$.

On the other hand, given the set W, as $K_{r+1}(t)$ we know there is some X_i containing at most $t - 1$ vertices joined to all of W.

Hence $|A| \leq \binom{n}{t}(t - 1)T^{r-1}$. Thus

$$T^r \binom{r\varepsilon n}{t} \leq \binom{n}{t}(t - 1)T^{r-1}$$

Now RHS$\leq \frac{n^t}{t!}(t - 1)T^{r-1}$, and LHS$\geq T^r \frac{1}{t!}(r\varepsilon n - t)^t$.

Thus $T^r(\frac{r\varepsilon n}{2})^t \leq n^t(t-1)T^{r-1}$, hence

$$T \leq (\frac{2}{r\varepsilon})^t(t-1)$$

Contradiction. $\qquad\square$

3.2 Hamiltonian Graphs

Definition. A *Hamilton cycle* in a graph G is a cycle of length $|G|$, i.e. going through all vertices of G.

If G has a Hamilton cycle we say G is *Hamiltonian*.

It's obvious that we can construct a graph that is not Hamiltonian as large as we want. Even more, the graph can have almost $\binom{n}{2}$ edges. A more interesting problem is, what if we condition on $\delta(G)$?

Theorem. (17, Dirac's theorem) Let $|G| = n \geq 3$ and $\delta(G) \geq \frac{n}{2}$. Then G is Hamiltonian.

Remark. This result is best possible: when n even we pick the disjoint union of two $K_{n/2}$, and for n odd we have two $K_{(n+1)/2}$ meeting only at one point.

Of course, there are Hamiltonian graphs of much smaller minimum degree, so this is not a necessary condition.

Proof. First, observe that G is connected: If, say $x \nsim y$, then $|\Gamma(x) \cup \Gamma(y)| \leq n-2$ but that sum is at least n, so they are not disjoint, i.e. there is a path from x to y.

Let $v_0, v_1, ..., v_k$ be a path of maximal length in G (length $k \leq n-1$). By maximality, $Gamma(v_0) \subset \{v_1, ..., v_k\}$, and the similar inclusion holds for $\Gamma(v_k)$. Let $A = \{i \in [k] : v_0 \sim v_i\}$ and $B = \{i \in [k] : v_k \sim v_{i-1}\}$. Then $|A \cup B| \leq k < n$ but $|A| + |B| \geq \frac{n}{2} + \frac{n}{2} = n$, hence $\exists i \in A \cap B$. So we have a cycle $C = v_0 v_1 ... v_{i-1} v_k v_{k-1} ... v_i v_0$ of length $k+1$. If $k = n-1$ then we've got a Hamilton cycle. If $k < n-1$, relabel the cycle as $C = u_0 u_1 ... u_k u_0$. By connectivity we have some $u_i \in C$ and $w \notin C$ with $w \sim u_i$. Then $w u_i ... u_k u_0 ... u_{i-1}$ is a path of length $k+1$. Contradiction. $\qquad\square$

Remark. The same proof gives the following: let $n > k \geq 3$ and let G be a connected graph with $\delta(G) \geq \frac{k}{2}$. Then G contains a path of length k, and a cycle of length at least $\frac{k+2}{2}$.

Konigsberg bridge problem seems superficially similar, but in fact much less interesting.

Definition. A *circuit* in a graph G is a sequence $v_0 ... v_k$ of vertices of G, not necessarily distinct, with $v_k = v_0$, if $1 \leq i \leq k$ then $v_{i-1} \sim v_i$ and each edges $v_{i-1} v_i$ and $v_{j-1} v_j$ are distinct. It is an *Euler circuit* if for every $e \in E(G)$ there is some i with $e = v_{i-1} v_i$.

If G has an Euler circuit, we say G is *Eulerian*. Apparently we are only interested in connected graphs (although we could have G not connected but Eulerian, with some degree zero vertices).

Proposition. (18)
Let G be a connected graph. Then G Eulerian $\iff$ $\forall v \in G$, $d(v)$ is even.

Proof. $\implies$ is obvious. For the other way, we do induction on G. The case $e(G) = 0$ is trivial. For $e(G) > 0$, let $v_0, ..., v_k = C$ be a circuit in G of maximal length. If C uses all edges of G then done. If not, delete all edges used in C to form H. In H, every vertex still has even degree, let H_1 be a component of H with at least one edge. By induction hypothesis H_1 has an Euler circuit D. Certainly C and D must meet at some vertex v, otherwise G is not connected. Join them at v to produce a longer circuit in G. $\qquad\square$

4 Graph Coloring

Recall map-colouring problem: how many colours do we need to colour the vertices of a graph that can be drawn with no crossings s.t. adjacent vertices get different colours?

Definition. A k-colouring of a graph G is a function $c : V(G) \to [k]$. In proofs, we often say 'blue', 'yellow' etc for 1,2,...

Note G has a k-colouring iff G is k-partite. So $\chi(G)$ is least number of colours needed to colour G.

4.1 Planar Graphs

Definition. Let $G = (V, E)$ be a graph. A drawing of G is an ordered pair (f, γ) where $f : V \to \mathbb{R}^2$ is an injection and $\gamma_i : E \to C([0, 1], \mathbb{R}^2)$ s.t.
(i) if $uv \in E$, then $\{\gamma(uv)(0), \gamma(uv)(1)\} = \{f(u), f(v)\}$;
(ii) if $e, e' \in E$ with $e \neq e'$, then $\gamma(e)((0, 1)) \cap (\gamma(e')((0, 1)) = \phi$;
(iii) if $e \in E$ then $\gamma(e)$ is injective; and
(iv) if $e \in E$ and $v \in V$, then $f(v) \notin \gamma(e)((0, 1))$.

However the above definition is not important at all. Intuitively vertices correspond to points, and edges corresponds to continuous curves between end vertices, with no unnecessary intersections. If G has a drawing, we say is *planar*.

We don't have to worry about topological problems (like space-filling curves) because it is known that any planar graph can be drawn with piecewise-linear edges (i.e. finitely many line segments).

Definition. Let G be a graph. A subdivision of G is a graph formed by repeated selecting $vw \in E(G)$, removing vw and adding vertex u and edges uv, uw.

Theorem. (19, Kuratowski's theorem)
Let G be a graph. Then G is planar iff G contains no subdivision of K_5 or $K_{3,3}$.

Definition. A leaf of a tree is a vertex of order 1.

Proposition. Every tree of order at least 2 has a leaf.

Proof. Let T be a tree, $|T| \geq 2$, and let $v_0...v_k$ be a path of max length in T. Now $v_k \sim v_{k-1}$. But v_k has no other neighbours in path (as T acyclic) and v_k has no other neighbours outside the path (by maximality). Hence v_k is a leaf. $\square$

Proposition. (21)
Let T be a tree, $|T| = n \geq 1$. Then $e(T) = n - 1$.

Proof. Do induction on n. Let $N = 1$. Then $e(T) = 0$. Now if $n > 1$, let v be a leaf. Then $T - v$ is also a tree with $|T - v| = n - 1$, so by induction hypothesis $e(T - v) = n - 2$. So $e(T) = n - 1$. $\square$

Proposition. (22)
Every tree is planar.

Proof. Let T be a tree, and $|T| = n$. Do induction on n. The case $n = 0, 1$ are obvious. Now if $n > 1$, Let $v \in T$ be a leaf. Now $T - v$ is planar. Let $u \in T$ be the neighbour of v. Now pick a small circle such that it only contains the centre u and some radii in drawing, without any other edges (this is always possible because the number of edges in $T - v$ is finite). Then we just place v at anywhere in the circle and we're done. $\square$

If we have a drawing of graph, it divides plane into connected regions called *faces*. Precisely one of these regions is unbounded, called the infinite face.

Theorem. (23, Euler's Formula)
Let G be a connected planar graph with $|G| = n \geq 1$, $e(G) = m$. Suppose G is drawn with l faces. Then $n - m + l = 2$.

Proof. Induction on m. If G is a tree: $m = n - 1, l = 1$ so we are ok. Otherwise, G has a cycle. Pick an edge e in the cycle and consider $G - e$. Then $|G - e| = n$, $e(G - e) = m - 1$. Moreover, in our drawing, removing e combines two faces, so we've drawn $G - e$ with $l - 1$ faces. So by induction hypothesis, $n - (m - 1) + (l - 1) = 2$. So $n - m + l = 2$. $\square$

Corollary. (24)
Let G be planar, $|G| = n \geq 3$. Then $e(g) \leq 3n - 6$.

Proof. Let $e(G) = m$. Draw G with l faces. WLOG suppose it is connected (else we add edges to it until it's connected). A special case is G has 3 vertices and 2 edges, but we're ok with that as well. Otherwise, we know $n - m + l = 2$. Each face has at least 3 edges in boundary of at most 2 faces. So $l \leq \frac{2m}{3}$. Thus $n - m + \frac{2}{3}m \geq 2$ so $m \leq 3n - 6$. $\square$

We can use this to get a result on chromatic number of planar graphs:

Proposition. (25)
Any planar graph is 6-colourable.

Proof. Let G be planar, $|G| = n$. We do an induction on n. For $n \geq 6$ it's trivial. For $n > 6$, by Corollary 24, $e(G) \leq 3n - 6$. Hence $\delta(G) \leq 5$. Now pick $v \in G$ with $d(v) \leq 5$. By induction hypothesis, $G - v$ can be 6-coloured. Therefore so can G. $\square$

A bit more work gives

Theorem. (26)
Any planar graph is 5-colourable.

Proof. Let G be planar, $|G| = n$. Do induction on n, and $n \leq 5$ is trivial. For $n > 5$, as in prop 25 we can find $v \in G$ with $\delta(v) \leq 5$ and $5-$colour $G - v$. If there is a colour missing in $\Gamma(v)$ then we can 5-colour G. Otherwise, draw G; wlog v has neighbours $x_1, ..., x_5$ in clockwise order around v with colours 1,...,5 respectively. Call a path in $G - v$ an ij-path if all its vertices have colour i or j. Given $x \in G - v$, the ij-component of x consists of those vertices reachable from x along ij-paths. If x_1, x_3 are in different 13-components then swap colours 1,3 on the 13-component of x_1 and give v colour 1. If not, then x_2, x_4 are in different 24-components (since the graph is planar) so swap colours 2,4 on 24-component of x_2 and give v colour 2. Then we can colour v accordingly. $\qquad\square$

Theorem. (27)
Any planra graph is 4-colourable.

Proof. Let G be planar, $|G| = n$. Induction on n. $n \leq 4$ is trivial. If $n > 4$, pick $v \in G$ with $d(v) \leq 5$, draw G, 4-colour $G - v$. If some colour missing on $\Gamma(v)$ then done. Otherwise there are 3 cases:
(1) $d(v) = 4$. Then similar argument would allow us to change the colour of some vertex in $\Gamma(v)$ and 4-colour G;
(2) $d(v) = 5$, say its neighbours are $x_1, ..., x_5$ clockwise, with colours $1, 2, 3, 4, 1$ respectively. If there's no 24-path from x_2 to x_4 then we are done. Otherwise, there can't be a 13-path from x_3 to x_1 or x_5, so we swap colour of the 13-component of x_3 and colour v with 3.
(3) $d(v) = 5$, with neighbours $x_1, ..., x_5$ of colours 1,2,3,1,4 clockwise respectively. If no 24-path from x_2 to x_5 then done; similarly, if no 34-path from x_3 to x_5 then we are done as well. Otherwise, swap colour 1,3 on 13-component of x_1, and swap colour 2,4 on 24-component of x_4. Then x_1 has turned colour 3 and x_4 has turned colour 2. Now we use colour 1 at v. So G is 4-colourable. $\qquad\square$

Remark. This proof is not examinable because it's rubbish.

4.2 General graphs

We know G is planar $\implies \chi(G) \leq 4$. What can we say about $\chi(G)$ is not given G planar?

Clearly $\chi(G)$ can be as large as we like, e.g. $\chi(X_k) = k$.

Lower bounds:
(a) If $K_k \subset G$ then $\chi(G) \geq k$.
The clique number of G is $\omega(G) = \max\{k : X_k \subset G\}$, then $\chi(G) \geq \omega(G)$. However, sometimes this bound is not good: we can find G with $\omega(G) = 2$ and $\chi(G)$ arbitrarily large (example sheet 2).

(b) Suppose we have a colouring of G. Let $W \subset V(G)$ be a colour class. Then W is an independent set – no edges with W. The *independence number* of G is $\alpha(G) = \max\{|W| : W \subset V(G), W \text{ independent}\}$. Then $\chi(G) \geq \frac{|G|}{\alpha(G)}$. Sometimes this bound is not good either: for example, take $G = K_k \cup \bar{K}_k$. Then $\chi(G) = k$, but $\frac{|G|}{\alpha(G)} = \frac{2k}{k+1} \approx 2$.

We also have a simple upper bound: list vertices in some order, colour each in turn using least colour not already used on its neighbours (Greedy algorithm). When we colour v, at most $d(v) \leq \Delta(G)$ colours unavailable. Thus $\chi(G) \leq \Delta(G) + 1$. (*)

But e.g. if $G = K_{1,t}$, then $\chi(G) = 2$, but $\Delta(G) = t$. We are interested in for which graphs G is (*) achieved. If G is complete, then $\chi(K_k) = k$, $\Delta(K_k) = k-1$; if G is an odd cycle (say a 5-cycle) then $\chi(G) = 3$, $\Delta(G) = 2$. That's all of them.

Theorem. (28, Brooks' Theorem)
Let G be a connected graph that is neither complete nor an odd cycle. Then $\chi(G) \leq \Delta(G)$.

Proof. We do induction on $|G|$. Write $\Delta = \Delta(G)$. We cannot have $\Delta = 0, 1$ as $G \not\cong K_1, K_2$. If $\Delta = 2$ then G is a path or an even cycle, so $\chi(G) = 2$. So we assume $\Delta \geq 3$.

Pick $v \in G$ and let H be a component of $G - v$. We have either $\Delta(H) < \Delta$, in which case by greedy algorithm bound we have $\chi(H) \leq \Delta(H) + 1 \leq \Delta$, or $\Delta(H) = \Delta$. Then H is connected and not odd cycle (as $\Delta \geq 3$). Moreover, $\exists u \in H$ with $u \sim v$ in G. In G, $d(u) \leq \Delta$ so in H, $d(u) \leq \Delta - 1$. So H is not regular, so not complete. Hence by induction hypothesis, $\chi(H) \leq \Delta$.

We do this for each component of $G - v$ to obtain a Δ-colouring c of $G - v$. If ther is a colour missing from $\Gamma(v)$, then use that colour at v. So assume that is not the case. So we have $\Gamma(v) = \{x_1, ..., x_\Delta\}$ with $\forall i c(x_i) = i$. We can also assume:

(i) if $i \neq j$ then there is an ij-path P_{ij} from x_i to x_j,

(ii) if $i \neq j$ then P_{ij} is entire ij-component containing x_i, x_j, and

(iii) if i, j, k distinct then P_{ij}, P_{ik} meet only at x_i.

(Why? If any of these fails then it is easy to check that the colouring c can be modified to change the colour of x_i (and allowing us to use colour i at v_i – not trivial, check).

As $G \not\cong K_{\Delta+1}$, there are some i, j with $i \neq j$, $x_i \not\sim x_j$. As $\Delta \geq 3$, pick $k \geq [\Delta] \setminus \{i, j\}$. Let u be the neighbour of x_i of colour j. Now swap colour i and k on the ik-component of x_i (i.e. on P_{ik}). This gives a new colouring c', with $c'(x_i) = k, c'(x_k = i)$. Also, if $w \in P_{ij}$ with $w \neq x_i$ then $c'(w) = c(w)$. So $c'(x_j) = c'(u) = j$. By (i) there is a kj-path from x_i to x_j, P'_{kj}. By (ii), $u \in P'_{kj}$. By (i), there is a ji-path P'_{ji} from x_j to x_k. By (ii), $u \in P'_{ji}$. But now P'_{kj} and P'_{ji} meet at u, which is not in the neighbourhood of v, so is not one of the x_i. This contradicts with (iii). $\square$

4.3 Graphs on surfaces

For a surface S, we define the chromatic number of S to be $max\{\chi(G)\}$ where G can be drawn on S.

W'll also use the Euler-Poincare Formula without proof: if G can be drawn with l faces, $|G| = n$, $e(G) = m$, then $n - m + l \geq E$.

Theorem. (Heawood's Theorem)
Let S be a closed boundaryless surface of Euler characteristic $E \leq 1$. Then

$\chi(S) \leq \lfloor \frac{7+\sqrt{49-24E}}{2} \rfloor.$

Proof. Let $\chi = \chi(S)$. Let G be a minimal $\chi-$chromatic graph that can be drawn on S -i.e. $\chi(G) = \chi$ but $H \subset G, H \neq G \implies \chi(H) \leq \chi - 1$. Clearly G is connected and $|G| \geq \chi$. Let $|G| = n \geq \chi$, $e(G) = m$ and draw G on S with L faces. Then by Euler-Poincare, $n - m + l \geq E$. As before, $l \leq \frac{2}{3}m$ so $n - \frac{1}{3}m \geq E$ so $m \leq 3n - 3E$. Hence $\delta(g) \leq \frac{2m}{n} \leq \frac{6n-6E}{n} = 6 - \frac{6E}{n}$ (*). On the other hand, if $v \in G$ then $G - v$ is $(\chi - 1)$-colourable and this colouring does not extend to a $(\chi - 1)$-colouring of G. So $d(v) \geq \chi - 1$. Hence $\delta(G) \geq \chi - 1$. Combining this with (*) we get, if $E \leq 0$: $\chi - 1 \leq \delta(G) \leq 6 - \frac{6E}{n} \leq 6 - \frac{6E}{\chi}$ as $n \geq \chi$. Hence $\chi^2 - 7\chi + 63 \leq 0$ and solve and we'll get the desired result; if $E = 1$, $\delta(G) \leq 6 - \frac{6}{n} < 6$ so $\delta(G) \leq 5$ Hence $\chi - 1 \leq 5$, so $\chi \leq 6$ which also satisfies the inequality. $\qquad\square$

Remark. Condition $E \leq 1$ rules out only the sphere. The theorem is true for sphere (4ct), but this proof doesn't work if $E = 2$.

4.4   Edge Colouring

A k-edge-colouring of a graph $G = (V, E)$ is a function $\phi : E \to [k]$ s.t. if $e, e' \in E$ with precisely one common vertex then $\phi(e) \neq \phi(e')$. The edge-chromatic number of G is the minimum k s.t. G has a k-edge-colouring, denoted $\chi'(G)$. Clearly $\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1$ by doing greedy. In fact,

Theorem. (30, Vizing's theorem)
Let G be a graph. Then $\chi'(G) \leq \Delta(G) + 1$.

Proof. Induction on $e(G)$. When $e(G) = 0$ it's trivial.
If $e(G) > 0$, let $\Delta = \Delta(G)$. Pick an edge xy_1. By induction hypothesis, we can find a $(\Delta + 1)$-edge-colouring of G-xy_1, ϕ, say. As $\Delta(G) < \Delta + 1$, there is some colour missing at each vertex. Let c_1 be missing at y_1. If c_1 is also missing at x then colour xy_1 by c_1. Otherwise, let $y_2 \in \Gamma(x)$ with $\phi(xy_2) = c_1$, and let c_2 be missing at y_2. We continue inductively(*): given distinct $y_1, ..., y_k \in \Gamma(x)$, distinct colours $c_1, ..., c_k$ missing at each y_i, and $\phi(xy_{i+1}) = c_i$. If c_k is missing at x, recolour xy_i in colour c_i and we are ok. Otherwise, let $y_{k+1} \in \Gamma(x)$ with $\phi(xy_{k+1}) = c_k$. Let c_{k+1} be missing at y_k. If $c_{k+1} \notin \{c_1, ..., c_k\}$ we continue as step (*); otherwise, assume wlog $c_{k+1} = c_1$ (if instead $c_{k+1} = c_j$ for some other j, we can uncolour xy_j, recolour xy_i in colour c_i $(1 \leq i < j)$ and relabel $y_j, y_{j+1}, ..$ as $y_1, y_2, ...$). Let c be a colour missing at x. Consider the cc_1-subgraph of G, i.e. only edges coloured c or c_1. This subgraph has max degree ≤ 2 so each component is a path or a cycle. Moreover, x, y_1, y_{k+1} have degree ≤ 1 in this subgraph. So x, y_1, y_{k+1} are not all in teh same component. If x, y_1 are in different components then swap c, c_1 on the component of y_1 and give xy_1 colour c. Otherwise, x, y_{k+1} are in different components. In this case, uncolour xy_{k+1} and recolour xy_i with colour i $(1 \leq i \leq k)$. Then swap colours c, c_1 on the cc_1 component of y_{k+1} and give xy_{k+1} colour c. $\qquad\square$

5   Connectivity

5.1   The Marriage Problem

Legal note: This problem was proposed in 1935, so marriage has 1 man and 1 woman.

Suppose we have n women and n men. For each women, we have a list of 'suitable' husbands. Can we marry everyone off?

We reformulate as a bipartite graph: X-women, Y-men, edge = suitable.

Definition. Let G be a bipartite graph with bipartition on (X, Y). A matching from X to Y is a set $M \subset E(G)$ s.t. for all $x \in X$, there is a unique $e \in M$ with $x \in e$, and for all $y \in Y$ there is at most one $e \in M$ with $y \in e$.

When can we find a matching? Obviously we need $\forall x \in X, |\Gamma(x)| \geq 1$. We also need if $x, y \in X$ distinct then $|\Gamma(x) \cup \Gamma(y)| \geq 2$. In general, if $A \subset X$ then we need $|\Gamma(A)| = |\cup_{x \in A} \Gamma(x)| \geq |A|$.

Surprisingly enough, this necessary condition is actually also sufficient.

Theorem. (31, Hall's Marriage Theorem)
Let G be a bipartite graph with bipartition (X, Y). Then G has a matching from X to Y iff G satisfies the Hall's condition: $\forall A \subset X, |\Gamma(A)| \geq |A|$.

Proof. It's obvious that this is a necessary condition.

To prove sufficiency, we go by induction on $|X|$. It's trivial for $|X| = 0, 1$.

For $|X| \geq 2$, an easy case is if $|\Gamma(A)| > |A|$ for all $A \subset X$ with $A \neq \phi, X$. Pick any $x \in X$. $|\Gamma(x)| \geq 1$, so pick $y \in \Gamma(x)$. Look at $G - \{x, y\}$. This graph satisfies Hall's condition, so has a matching from $X - \{x\}$ to $Y - \{y\}$. Add edge xy.

The harder case is when $|\Gamma(A)| = |A|$ for some $A \neq \phi, X$. So we can split the bipartite graph into two parts, $A \cup \Gamma(A)$, and $X \setminus A$ and $Y \setminus \Gamma(A)$. Now for $A \cup \Gamma(A)$ we can find a matching between them by induction hypothesis; for its complement, take $B \subset X \setminus A$. Then $|\Gamma_{G_2}(B)| = |\Gamma(A \cup B) \setminus \Gamma(A)| = |\Gamma(A \cup B)| - |\Gamma(A)| \geq |A \cup B| - |A| = |B|$. Hence the complement also satisfies Hall's condition and has a matching from $X \setminus A$ to $Y \setminus \Gamma(A)$. Combine the two matchings we get a matching from X to Y in G. $\qquad \square$

Definition. Let $G = (V, E)$ be a graph. A set $F \subset E$ is *independent* if no two edges in F share a vertex.

Note: a matching from X to Y is just a set of $|X|$ independent edges.

Corollary. (32, Defect Hall)
Let G be bipartite with bipartition (X, Y), and let $d \geq 1$. Then G contains $|X| - d$ independent edges iff for all $A \subset X, |\Gamma(A)| \geq |A| - d$.

Proof. We only have to prove backward. In marriage terminology: introduce d imaginary perfect men, suitable husbands for all the women. This satisfies Halls' condition, so has matching from X to Y. So at most d women unmarried. $\quad\square$

Corollary. (33, Polyandrons Hall)
Let G be a bipartite graph, bipartition (X, Y), $d \geq 2$. Then G contains a set of $d|X|$ edges, each vertex in X in precisely d of them, each vertex in Y in at most one $\iff \forall A \subset X, |\Gamma(A)| \geq d|A|$.

Proof. We only have to prove backward as well. In marriage terminology, we clone each women $d - 1$ times, so we have d copies of each. This satisfies Hall's condition, so have matching from X to Y. To finish up we just destroy the clones. $\quad\square$

Further extensions are possible, e.g.
1) (Variable polyandrons): Women $x_1, ..., x_n$; numbers $d_1, ..., d_n \geq 1$. Give woman x_i d_i husbands.
2) (Defect polyandrons): Aim to give each women d husbands but allow d 'missing' husbands.
3) (Harder) (Same sex) If G is a graph, we can find a 1-factor in G – i.e. a set of $\frac{|G|}{2}$ independent edges?

5.2   Connectivity

Let $k \geq 1$. We say a graph G is k-connected if whenever $W \subset V(G)$ with $|W| < k$, then $G - W$ is connected (Assume for now that G is not a complete graph).

For example, 1-connected = connected.

G is 2-connected = G is connected and has no 'cutvertex', i.e. a vertex that, if removed, would disconnect the graph.

Definition. Let G be a graph and $a, b \in V(G)$ be distinct. A collection of paths from a to b is *independent* if the paths meet only at a and b.

Suppose for all distinct $a, b \in G$ there are k independent paths from a to b. Then G is k-connected.

Converse? If G is k-connected and can only find $k - 1$ independent ab-paths for some $a, b \in G$ then would like to say that 'delete one vertex from each path to disconnect the graph, contradiction?' But this doesn't work.

In fact, the converse is true, but is not obvious.

Definition. Let G be a graph and $A, B \subset V(G)$. An $AB-$path is a path that meets A in its vertex and nowhere else, and meets B in its last vertex and nowhere else.

A set $W \subset V(G)$ is an $AB-$separator if $G - W$ contains no $AB-$path.

Remark. • Both A and B are $AB-$seperator.
• Suppose G is k-connected. Then as long as $|A|, |B| \geq k$, we have that any $AB-$separator W satisfies $|W| \geq k$.
• Note we do not require $A \cap B = \phi$. If $v \in A \cap B$ then v is an AB-path (of length 0). So any AB-separator necessarily contains $A \cap B$.

Theorem. (34)
Let G be a graph and $A, B \subset V(G)$. Let $k = \min\{|W| : W$ is an AB-separator$\}$. Then G contains k vertex-disjoint AB-paths.
Note that vertex disjoint means paths don't even meet inside A or B.

We'll prove this next time.

Corollary. (35, Menger's Theorem)
Let G be an incomplete k-connected graph and let $a, b \in V(G), a \neq b$. Then G contains k independent ab-paths.

Proof. Suppose first $a \nsim b$. Let $A = \Gamma(a)$ and $B = \Gamma(b)$. We have k-connected and $G - A, G - B$ disconnected, so $|A|, |B| \geq k$. Hence any AB-separator W has $|W| \geq k$, so by theorem 34 there are k vertex-disjoint AB paths. Extend these to a, b. If instead $a \sim b$, $G - ab$ is $(k-1)$-connected so has $k - 1$ independent ab paths by first part. ab is another one (of length 1). $\square$

Definition. If G is an incomplete graph, the *connectivity* of G is

$$\mathcal{K}(G) = \max(\{k \geq 1 : G \text{ is k-connected}\} \cup \{0\})$$

In light of Menger, we define $\mathcal{K}(K_n) = n - 1$ for $n \geq 2$.

Proof. (of theorem 34)
We do induction on $e(G)$. When $e(G) = 0$, the smallest AB-separator is $A \cap B$, and each vertex of $A \cap B$ gives an AB-path (of length zero).
When $e(G) > 0$, pick $xy \in E(G)$. Let W be an AB-separator of minimum order in $G - xy$. If $|W| \geq k$ then by induction hypothesis there are k vertex-disjoint AB paths in $G - xy$ and so also in G. So assume $|W| < k$. Then $W \cap \{x\}$ is an AB-separator in G. Hence $|W \cap \{x\}| \geq k$. So $|W| \geq k - 1$, so $|W| = k - 1$. Write $W = \{w_1, w_2, ..., w_{k-1}\}$. As $|W| < k$, $G - W$ contains an AB-path. This path must include the edge xy. Assume wlog x comes before y on this path (if not, swap x and y. Let $X = W \cap \{x\}$ and $Y = W \cap \{y\}$. Let U be an AX-separator in $G - xy$. Then U is an AB-separator in G, so $|U| \geq k$. So by induction hypothesis, we have k vertex-disjoint AX-paths in $G - xy$, say $P_0, P_1, ..., P_{k-1}$ ending at $x, w_1, ..., w_{k-1}$ respectively. Similarly, there are k vertex-disjoint YV-paths in $G - xy$, say $Q_0, Q_1, ..., Q_{k-1}$ starting at $y, w_1, ..., w_{k-1}$ respectively. Given paths $P = u_0 u_1 ... u_l$ and $Q = v_0 v_1 ... v_m$ meeting only at $u_l = v_0$, write $p \vee Q$ for the path $u_0 ... u_l v_1 ... v_m$. Then $P_0 \vee xy \vee Q_0$ and $P_i \vee Q_i$ $(1 \leq i \leq k - 1)$ are k vertex-disjoint AB-paths in G. $\square$

In fact, Hall's Marriage Theorem is also a consequence of T34. Let G be a bipartite graph with bipartition $(X < Y)$ s.t. $\forall A \subset X, |\Gamma(A)| \geq |A|$. Let W be

an XY-separator. Then $\Gamma(X \setminus W) \subset Y \cap W$. So $|W| = |X \cap W| + |Y \cap W| \geq$ $|X \cap W| + \Gamma(X \setminus W)| \geq |X \cap W| + |X \setminus W| = |X|$. Hence by theorem 34, we have $|X|$ vertex-disjoint XY-paths, i.e. a matching from X to Y.

5.3   Edge-connectivity

Definition. Let G be a graph with $|G| \geq 2$ and let $l \geq 1$. We say G is l-edge-connected if whenever $D \subset E(G)$ with $|D| < l$ we have G-D connected. The *edge-connectivity* of G is

$$\lambda(G) = \max(\{l \geq 1 : G \text{ is } l\text{-edge-connected}\} \cup \{0\})$$

Corollary. (36, Edge Menger)
Let G be l-edge connected, and $a, b \in V(G)$ be disjoint. Then G has l edge-disjoint ab-paths.

Proof. If $G = K_2$ it's trivial. Suppose now $G \neq K_2$. The line graph of $G = (V < E)$ is the graph $L(G) = (E, F)$ where $F = \{ee' : e, e' \in E, e, e' \text{ share precisely 1 vertex}\}$. Then $L(G)$ is l-connected. Pick $a', b' \in E$ with $a \in a'$, $b \in b'$, $a' \neq b'$. By Menger, $L(G)$ has l independent $a'b'$−paths. This yields l edge-disjoint ab-paths in G. $\square$

There's a small mistake. We need to change 'Let $a', b' \in E$ be distinct with $a \in a'$, $b \in b'$ to be 'Add extra vertices a', b' to $L(G)$ with a' joined to each $e \in E$ with $a \in e$ and b' joined to each $e \in E$ with $b \in e$.

6 Probabilistic techniques

6.1 The probabilistic method

Recall the *Ramsey number* $R(s)$ is the least n s.t. if we colour K_n with edges blue/yellow, then there exists a monochromatic K_s. In Chapter 1 we proved that $R(s) = O(4^s)$, and in example sheet we knew $R(s) = \Omega(s^3)$. Similarly but much harder we can prove $R(s) = \Omega(s^4)$. Remarkably,

Theorem. (37, Erdos)
$R(s) = \Omega(\sqrt{2}^s)$.

Proof. Fix n, s. Colour each edge of K_n blue/yellow at random independently, each colour equally likely. Given $H \subset K_n$ with $H \cong K_s$, $\mathbb{P}(H_s \text{ monochromatic}) = 2\mathbb{P}(H \text{ blue}) = 2(\frac{1}{2})^{\binom{s}{2}}$. So $\mathbb{P}(K_n \text{ has a monochromatic } K_s) \leq \binom{n}{s}2(\frac{1}{2})^{\binom{s}{2}} \leq \frac{n^s}{s!}2(\frac{1}{2})^{\binom{s}{2}} \leq n^s 2^{-\frac{s(s-1)}{2}} = (\frac{n}{\sqrt{2}^{s-1}})^s < 1$ if $n < \sqrt{2}^{s-1}$. So if $n < \sqrt{2}^{s-1}$ then there is some colouring with no monochromatic K_s. So $R(s) \geq \sqrt{2}^{s-1}$. $\square$

6.2 Modifying a random graph

In theorem 37, we can randomly colour K_n blue/yellow and find $\mathbb{P}(\text{no monochromatic } K_s) > 0$. Hence $\exists$ colouring with no monochromatic K_s. In general, we do something random which lead to the probability that the event we desired happens is positive (*), then we can get a lower bound.

However, sometimes it's too hard to prove (*) directly and we have to work a bit harder. Instead, we can have the probability of thing we want 'almost happens' being positive, then take an example where the thing 'almost happens', and modify it somehow to get what we want.

Recall the Problem of Zarankiewicz: we defined in chapter 2, $z(n,t)$ to be the greatest possible number of edges in a graph with n vertices in each class containing no $K_{t,t}$. We proved upper bound: if $t \geq 2$, $z(n,t) = O(n^{2-1/t})$. How about lower bounds?

Theorem. (38)
If $t \geq 2$, then $z(n,t) = \Omega(n^{2-\frac{2}{t+1}})$.

Proof. Strategy: given n, p, let G be a random bipartite graph with vertex classes X, Y, $|X| = |Y| = n$, where for each $x \in X$, $y \in Y$, there is a chance p that xy is an edge of G independently. Let $A = e(G)$ and let B be the number of $K'_{t,t}s$ in G, so A, B are random variables. Aim: given n, try to choose p s.t. $\mathbb{E}(A - B)$ large, specifically $\mathbb{E}(A - b) = \Omega(n^{2-\frac{2}{t+1}})$. Then we can find a specific graph G with $A - B = \Omega(n^{2-\frac{2}{t+1}})$. Remove an edge from each $K_{t,t}$ in G to form a graph H with no $K_{t,t}$ and $e(H) = \Omega(n^{2-\frac{2}{t+1}})$. Now $\mathbb{E}A = n^2p$, and $\mathbb{E}B = \binom{n}{t}^2 p^{t^2} \leq \frac{1}{t!^2}n^{2t}p^{t^2}$,

so $\mathbb{E}(A - B) \geq n^2 p - \frac{1}{t!^2} n^{2t} p^{t^2}$. We want $n^2 p = n^{2t} = p^{t^2}$. So we just pick $p = n^{\frac{2-2t}{t^2-1}} = n^{-\frac{2}{t+1}}$. Then $\mathbb{E}(A - B) \geq (1 - \frac{1}{t!^2}) n^{2 - \frac{2}{t+1}}$ as required. $\qquad\square$

Recall in example sheet 2 we constructed a triangle free graph of large chromatic number. In fact:

Theorem. (39)
Let $g \geq 3$, $k \geq 2$. Then there is a graph G with no cycles of length $\leq g$ and $\chi(G) \geq k$.

Proof. Strategy: fix n and p. Let G be a random graph with n vertices, each possible edge present independently with probability p. Let X be the number of short cycles in G. Recall that $\chi(G) \geq \frac{|G|}{\alpha(G)}$ where $\alpha(G)$ is independence number of G. For cycles, by 'short' we mean of length $\leq g$. Aim: pick n and p such that (1) $\mathbb{P}(X > \frac{n}{2}) < \frac{1}{2}$, and (2) $\mathbb{P}(\alpha(G) \geq \frac{n}{2k}) < \frac{1}{2}$. Then $\mathbb{P}(X > \frac{n}{2}$ or $\alpha(G) \geq \frac{n}{2k}) < 1$, so we can pick a specific G such that $X \leq \frac{n}{2}$ and $\alpha(G) < \frac{n}{2k}$. Remove a vertex from each short cycle to get H with $|H| \geq \frac{n}{2}$ and $\alpha(H) < \frac{n}{2k}$ so $\chi(H) > \frac{n/2}{n/2k} = k$.

(1) For $3 \leq i \leq g$, let X_i be the number of cycles of length i appearing as subgraphs of G. Then $\mathbb{E}X_i = \binom{n}{i}\frac{i!}{i2}p^i \leq (np)^i$. Now $X = \sum_{i=3}^{g} X_i$, so $\mathbb{E}X_i \leq \sum_{i=3}^{g}(np)^i < g(np)^g$ as long as $np \geq 1$ (*). By Markov inequality, $\mathbb{P}(X > \frac{n}{2}) \leq \frac{\mathbb{E}X}{n/2} < 2gn^{g-1}p^g \leq \frac{1}{2}$ if $p \leq (\frac{1}{4g})^{1/g}n^{1/g-1}$. Take $p = (\frac{1}{4g})^{1/g}n^{1/g-1}$. Then $np = (\frac{1}{4g})^{1/g}n^{1/g} \geq 1$ if n is sufficiently large, satisfying (*).

(2)
$$\mathbb{P}(\alpha(G) \geq \frac{n}{2k}) \leq \binom{n}{n/2k}(1-p)^{\binom{n/2k}{2}}$$
$$\leq n^{n/2k}e^{-p\frac{n^2}{16k^2}}$$
$$= \exp\{\frac{n}{2k}\log n - \frac{n^2}{16k^2}\cdot(\frac{1}{4g})^{1/g}n^{1/g-1}\} \to 0$$

as the negative term in exp dominates when $n \to \infty$. $\qquad\square$

Properties of Expectation:
We used:
1. Lineartiy of expectation:
$\mathbb{E}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathbb{E}X_i$.

e.g. we had G bipartite, n vertices in each class, each possible edge present independently with probability p; let B be the number of $K_{t,t}$'s in G, we said $\mathbb{E}B = \binom{n}{t}^2 p^{t^2}$. Why?

Let $H_1,...,H_N$ be the possible $K_{t,t}$'s ($N = \binom{n}{t}^2$). Let $X_i = 1$ if H_i appears in G, and 0 otherwise. So $\mathbb{E}X_i = p^{t^2}$. Then $B = \sum_{i=1}^{N} X_i$, so $\mathbb{E}B = \sum_{i=1}^{N} \mathbb{E}X_i = N\mathbb{E}X_i = \binom{n}{t}^2 p^{t^2}$ as desired.

2. Markov's Inequality:
If X is a non-negative r.v. with finite mean and $\lambda > 0$, then

$$\mathbb{P}(X \geq \lambda) \leq \frac{\mathbb{E}X}{\lambda}$$

To prove that, we have $\lambda 1_{\{X \geq \lambda\}} \leq X$. Take expectations, we get $\lambda \mathbb{P}(X \geq \lambda) \leq \mathbb{E}X)$.

3. Chebyshev's Inequality:
Let X be a r.v. with finite mean μ and finite variance σ^2. Then $\mathbb{P}(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon}$ for all $\varepsilon > 0$. To prove this, apply Markov to the r.v. $(X - \mu)^2$ with mean σ^2.

6.3 The Structure of Random Graphs

Given $n \geq 1$ and $p \in [0,1]$, the probability space $\mathcal{G}(n,p)$ consists of all graphs G with vertex set $\{1, 2, ..., n\}$ where for $i, j \in [n]$ distinct, $P(ij \in E(G)) = p$, independently for different edges. What does a typical $G \in \mathcal{G}(n,p)$ look like? For example, does G contains a triangle? Intuitively the answer should be usually yes if p is 'large', and usually no if p is 'small'. In fact, we have a 'sharp threshold' for the answer.

First we consider an easier example: does G contain an edge? Let $G \in \mathcal{G}(n,p)$ and let $X = e(G)$. Let $\mu = \mathbb{E}X = \binom{n}{2}p = N_p$, where $N = \binom{n}{2}$. By Markov, $\mathbb{P}(G$ has an edge$) = \mathbb{P}(X \geq 1) \leq \mathbb{E}X = N_p$. So if $\mu = N_p \to 0$ as $n \to \infty$, then $\mathbb{P}(G$ has an edge$) \to 0$, i.e. if $p = o(\frac{1}{N}) = o(n^{-2})$, then $\mathbb{P}(G$ has an edge$) \to 0$.

What if p is bigger, say $p = \omega(n^{-2})$, in which case $\mu = N_p \to \infty$? We nned to think about variance. What is $Var(X)$?
Let $e_1, e_2, ..., e_N$ be the possible edges of G $(N = \binom{n}{2})$, and let $X_i = 1$ if e_i appears in G, and 0 if not. Then $X = \sum_{i=1}^{N} X_i$. Now $\mathbb{E}X_i = p$, and $\mathbb{E}X_i^2 = p$, so $Var(X_i) = \mathbb{E}X_i^2 - (\mathbb{E}X_i)^2 = p - p^2$. The X_i's are independent, so $Var(X) = \sum_{i=1}^{N} Var(X_i) = N(p - p^2) = \sigma^2$, say.

By Chebyshev, $\mathbb{P}(X = 0) \leq \mathbb{P}(|x - \mu| \geq \mu) \leq \frac{\sigma^2}{\mu^2} = \frac{N(p-p^2)}{(Np)^2} = \frac{1}{Np} - \frac{1}{N} = \frac{1}{\mu} - \frac{1}{N} \leq \frac{1}{\mu} \to 0$ as $n \to \infty$. So if $p = \omega(n^{-2})$, then $\mathbb{P}(X = 0) \to 0$. So $\mathbb{P}(X \geq 1) \to 1$. In other words, $\mathbb{P}(G$ has an edge$) \to 1$.

We have proved that $p = n^{-2}$ is a 'sharp threshold' for $G = \mathcal{G}(n,p)$ to have an edge, in the sense that:
• If $p = o(n^{-2})$, then almost every $G \in \mathcal{G}(n,p)$ has no edge, whereas
• If $p = \omega(n^{-2})$ then almost every $G \in \mathcal{G}(n,p)$ has an edge.

Here we say 'almost every $G \in \mathcal{G}(n,p)$ has property Q' to mean $\mathbb{P}(G \in \mathcal{G}(n,p)$ has $Q) \to 1$ as $n \to \infty$. Note that the term 'almost every' has some different meaning in some other course.

More generally, let $A_1, ..., A_n$ be events, X is the number of A_i that occur. Then

$X = \sum_{i=1}^{n} X_i$ where $X_i = 1$ if A_i occurs, and 0 if not. So $\mathbb{E}X_i = \mathbb{P}(A_i)$. Hence by linearity of expectation, $\mathbb{E}X = \sum_{i=1}^{n} \mathbb{P}(A_i)$.

We know $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$. Now

$$\mathbb{E}X^2 = \mathbb{E}[(\sum_{i=1}^{n} X_i)^2]$$

$$= \mathbb{E}[(\sum_{i=1}^{n} X_i)(\sum_{j=1}^{n} X_j)]$$

$$= \mathbb{E}\sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}(X_i X_j)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{P}(A_i \cap A_j)$$

and $(\mathbb{E}X)^2 = (\sum_{i=1}^{n} \mathbb{P}(A_i))^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{P}(A_i)\mathbb{P}(A_j)$. Hence

$$Var(X) = \sum_{i=1}^{n}\sum_{j=1}^{n} (\mathbb{P}(A_i \cap A_j) - \mathbb{P}(A_i)\mathbb{P}(A_j))$$

Note that, if A_i, A_j are indepndent, then the term in the sum is 0. So independent pairs do not contribute to this sum.

Proposition. (40)
$p = 1/n$ is a sharp threshold for $G \in \mathcal{G}(n, p)$ to contain a triangle, in the similar sense as the above for an edge.

Proof. Let $G \in \mathcal{G}(n, p)$ and let X be the number of triangles in G. Let $\mu = \mathbb{E}X$ and $\sigma^2 = Var(X)$. Then $\mu = \binom{n}{3}p^3 \sim \frac{1}{6}(np)^3$. Also, $\sigma^2 = \binom{n}{3}(p^3 - p^6) + \binom{n}{3}3(n-3)(p^5 - p^6) \leq n^3p^3 + n^4p^5$.

Suppose first $p = o(\frac{1}{n})$, i.e. $np \to 0$. Then by Markov, $\mathbb{P}(X = 0) = 1 - \mathbb{P}(X \geq 1) \geq 1 - \mu \to 1$ as $n \to \infty$. Suppose instead $p = \omega(\frac{1}{n})$ so $np \to \infty$. Then by Chebyshev,

$$\frac{\sigma^2}{\mu^2} \leq \frac{1}{\mu^2}(n^3p^3 + n^4p^5) \sim \frac{36}{n^6p^6}(n^3p^3 + n^4p^5) = \frac{36}{(np)^3} + \frac{36}{n \cdot np} \to 0$$

as $n \to \infty$. $\square$

Following the same idea we should be able to find this threshold for any fixed graph, the only problem is just that the variance calculation is much harder.

Another question is, what is the clique number $\omega(G)$ for typical $G \in \mathcal{G}(n, p)$?

Theorem. (41)
There exists a function $d : \mathbb{N} \to \mathbb{N}$ such that a.e. $G \in \mathcal{G}(n, 1/2)$ has $\omega(G) \in \{d-1, d, d+1\}$ (where $d = d(n)$).

Proof. (sketch)
Let $G \in \mathcal{G}(n, 1/2)$. Given k, let X_k be the number of K_k's in G. Then $\mathbb{E}X_k = \binom{n}{k} 2^{-\binom{k}{2}} = g(k)$, say. We seek d with $g(d) \approx 1$. If $1 \ll k \ll n$, then $g(k) \approx \frac{1}{k!} n^k 2^{-k^2/2}$ so $\log_2 g(k) \approx k \log_2 n - \log_2 k! - k^2/2$. We want $g(d) = 1$ so $\log_2 g(d) \approx 0$ so $d \approx 2 \log_2 n$. In fact, we can find $d \sim 2 \log_2 n$ s.t. $g(d) \geq 1$ and $g(d+1) < 1$.

Suppose $k \sim 2 \log_2 n$. Then

$$\frac{g(k+1)}{g(k)} = \frac{n! k! (n-k)!}{(k+1)!(n-k-1)! n!} 2^{\binom{k}{2} - \binom{k+1}{2}}$$
$$= \frac{n-k}{k+1} 2^{-k}$$
$$\leq n 2^{-k} \approx \frac{1}{n} \to 0$$

as $n \to \infty$. Thus $g(d+2) \to 0$ and $g(d-1) \to \infty$ as $n \to \infty$.

We must show:
1. $\mathbb{P}(X_{d+1} = 0) \to 1$ as $n \to \infty$, and
2. $\mathbb{P}(X_{d-1} = 0) \to 0$ as $n \to \infty$.

1.
$$\mathbb{P}(X_{d+2} = 0) = 1 - \mathbb{P}(X_{d+2} \geq 1)$$
$$\geq 1 - \mathbb{E}X_{d+2}$$
$$= 1 - g(d+2) \to 1$$

as $n \to \infty$ (by Markov).

2. Let $\mu = \mathbb{E}X_{d-1}$ and $\sigma^2 = Var(X_{d-1})$. Note $\mu = g(d-1) \to \infty$. Now

$$\sigma^2 = \sum_{i=2}^{d-1} \binom{n}{d-1}\binom{d-1}{i}\binom{n-(d-1)}{(d-1)-i} \left[2^{-(2\binom{d-1}{2} - \binom{i}{2})} - 2^{-2\binom{d-1}{2}} \right]$$

Hence

$$\frac{\sigma^2}{\mu^2} \leq \sum_{i=2}^{d-1} \frac{\binom{d-1}{i}\binom{n-d+1}{d-1-i}}{\binom{n}{d-1}} 2^{\binom{i}{2}} = \sum_{i=2}^{d-1} \Delta_i$$

where Δ_i is the above summand.

Now

$$\Delta_{d-1} = \frac{1}{\binom{n}{d-1}2^{-\binom{d-1}{2}}} = \frac{1}{\mu} \to 0,$$

$$\Delta_{d-2} = \frac{(d-1)(n-d+1)}{\binom{n}{d-1}}2^{\binom{d-2}{2}}$$

$$= \frac{(d-1)(n-d+1)}{\binom{n}{d-1}2^{-\binom{d-1}{2}}}2^{-(d-2)}$$

$$\leq \frac{4nd2^{-d}}{\mu}$$

$$\approx \frac{4n \cdot 2\log_2 n \cdot n^{-2}}{\mu} = \frac{8\log_2 n/n}{\mu} \to 0,$$

$$\Delta_2 = \frac{\binom{d-1}{2}\binom{n-d+1}{d-3}}{\binom{n}{d-1}} \cdot 2$$

$$\leq \frac{2\binom{d-1}{2}\binom{n}{d-3}}{\binom{n}{d-1}}$$

$$= \frac{(d-1)(d-2)(d-1)(d-2)}{n(n-1)}$$

$$\sim \frac{d^4}{n^2} \sim \frac{16(\log_2 n)^4}{n^2} \to 0$$

It can be shown that these three terms dominate: the contribution to the sum from $\Delta_3, ..., \Delta_{d-3}$ is negligible. Thus $\frac{\sigma^2}{\mu^2} \to 0$.

Now $\mathbb{P}(X_{d-1} = 0) \leq \mathbb{P}(|X_{d-1} - \mu| \geq \mu) \leq \frac{\sigma^2}{\mu^2} \to 0$ by Chebyshev. $\qquad\square$

Remark. 1. This also shows a.e. $G \in \mathcal{G}(n, 1/2)$ has $\omega(G) \sim 2\log_2 n = \frac{2\log n}{\log 2}$.
2. Same proof works to give same result for general p. Then

$$\omega(G) \sim \frac{2\log n}{\log \frac{1}{p}}$$

Corollary. (42)
a.e. $G \in \mathcal{G}(n, p)$ has $\chi(G) \leq (1 + o(1))\frac{n\log 1/q}{2\log n}$, where $q = 1 - p$.

Proof. Let $G \in \mathcal{G}(n, p)$. Then $\bar{G} \in \mathcal{G}(n, q)$. So by theorem 41, with probability tending to 1 as $n \to \infty$, we have $\omega(\bar{G}) \sim 2\log n\log(1/q)$. So $\alpha(G) \sim \frac{2\log n}{\log(1/q)}$, so $\chi(G) \geq \frac{|G|}{\alpha(G)} \sim \frac{n\log(1/q)}{2\log n}$. $\qquad\square$

7 Algebraic methods

Let G be a connected graph, $u, v \in G$. The distance from u to v is $d(u, v)$, the length of the shortest path from u to v. The diameter of G is $\max_{u,v \in G} d(u, v)$.

Graphs with diameter 1 are just K_n.

For diameter 2 we have many more possibilities. Question: How many vertices can a graph of diameter 2 have if say $\Delta(G) = k$? If we fix $x \in G$, then $V(G) = \{x\} \cup \Gamma(x) \cap \Gamma(\Gamma(x))$. So $|G| \le 1 + k + k^2 - k = k^2 + 1$. Can we actually achieve this?

Definition. A *Moore graph* is a graph G s.t. for some k, $|G| = k^2 + 1$, $\Delta(G) = k$, diameter of G is 2.

For which k can we find a Moore graph?

In general, if $u, v \in G$ are distinct (G Moore), if $u \sim v$ then u, v have no common neighbours, i.e. no triangles; if $u \not\sim v$, then u, v has exactly one common neighbour, so no 4-cycles.

7.1 The Chromatic Polynomial

Suppose G is a k-colourable graph. How many k-colourings does G have?

Example. $G = K_n$, $k < n$: 0; $k \ge n$: $k(k-1)...(k-n+1)$.

G a tree: can repeatedly remove leaves until we get down to a single vertex. Reverse this order and do greedy: $k(k-1)^{|G|-1}$.

Write $f_G(k)$ for the number of k-colourings of G.

Definition. Let G be a graph and $e = uv \in E(G)$. The *contraction* of G over e is the graph G/e formed from G by deleting vertices u, v, adding a new vertex e^* with $\Gamma(e^*) = \Gamma(u) \cup \Gamma(v)$.

Theorem. (43, cut-fuse relation)
Let G be a graph, $e \in E(G)$, $k \ge 1$. Then $f_G(k) = f_{G-e}(k) - f_{G/e}(k)$.

Proof. Let $e = uv$. Let c be a k-colouring of $G - e$. If $c(u) \ne c(v)$ then c is a k-colouring of G, and every k-colouring of G arises uniquely like this. If $c(u) = c(v)$, then c yields a k-colouring of G/e, and every k-colouring of G/e arises uniquely like this. $\square$

Corollary. (44)
Let G be a graph. Then f_G is a polynomial.

Proof. Induction on $e(G)$. if $e(G) = 0$ we have $f_G(k) = k^{|G|}$. If $e(G) > 0$, pick $e \in E(G)$. Then $f_{G-e}, f_{G/e}$ are polynomials by induction hypothesis, and hence f_G as their sum is a polynomial. $\square$

In light of this, we call f_G the *chromatic polynomial* of G.

The original motivation is to study the four-colour theorem. We can rephrase the 4CT as If G is planar, then $f_G(4) \neq 0$. The idea is to do something algebraic to prove this. The aim was to prove if G is planr and α is a root of f_G then $\alpha < 4$. Unfortunately, no one managed to do this. However, f_G carries some information about G. For example:

Corollary. (45)
If $|G| = n$, $e(G) = m$, then $f_G(X) = X^n - mX^{n-1} + ...$, i.e. the top two coefficients depend completely on $|G|$ and $e(G)$.

Proof. Induction on $e(G)$. When $e(G) = 0$, clearly $f(X) = X^n$. When $e(G) > 0$, pick $e \in E(G)$. Then $f_G(X) = f_{G-e}(X) - f_{G/e}(X) = X^n - (m-1)X^{n-1} + ... - X^{n-1} - ...$. So done. $\qquad\square$

7.2 Eigenvalues

Definition. Let G be a graph with $V(G) = \{1, 2, ..., n\}$. The *adjacency matrix of G* is the $n \times n$ matrix A where $A_{ij} = 1$ if $i \sim j$, and is 0 otherwise.

Note that the adjacency matrix is always real symmetric. Also it has 0 on its diagonal. Now what is A^2? We have $(A^2)_{ij} = A_{ik}A_{kj} = |\Gamma(i) \cap \Gamma(j)|$, the number of common neighbours of i, j. In particular, $(A^2)_{ii} = d(i)$. So $\operatorname{tr}(A^2) = 2e(G)$. More generally, $(A^l)_{ij}$ is the number of ways to walk from i to j in l steps. We define a *walk* of length l from u to v to be a sequence $u = u_0...u_l = v$ of (not necessarily distinct) vertices with $u_{i-1} \sim u_i$ for $1 \leq i \leq l$. So we can say $(A^l)_{ij}$ is the number of walks of length l from i to j.

If G is a graph, the *eigenvalues* of G are the eigenvalues of its adjacency matrix A.

Note that change of labelling of vertices is effectively a reordering of basis, which doesn't affect eigenvalues. Also, A is real symmetric, so the eigenvalues are real, A is diagonalizable, and we have a orthonormal basis of eigenvectors.

Sometimes it's easy to spot eigenvalues/eigenvectors.

Example. if $G = K_n$, then $(1, 1, ..., 1)^T$ is an eigenvector with eigenvalue $n - 1$. Also $A + I$ is rank 1 so nullity $n - 1$, so -1 is an eigenvalue with multiplicity $n - 1$.

Example. If $G = C_4$, then A has rank 2 so nullity 2. So 0 is an eigenvalue with multiplicity 2. Also we note that 2 is an eigenvalue with eigenvector $(1, 1, 1, 1)$, and -2 is an eigenvalue with eigenvector $(1, -1, 1, -1)$.

Let A have orthonormal basis $e_1, ..., e_n$ of eigenvectors with eigenvalues $\lambda_1 \geq \lambda - 2 \geq ... \geq \lambda_n$. Given $x \in \mathbb{R}^n$, we can write $x = \sum_{i=1}^n x_i e_i$, where $\sum_{i=1}^n x_i^2 = ||x||^2$. If we let A act on this, we get $Ax = \sum_{i=1}^n \lambda_i x_i e_i$, so $Ax \cdot x = \sum_{i=1}^n \lambda_i x_i^2$. Hence $\lambda_1 = \max ||x|| = 1 Ax \cdot x$, and $\lambda_n = \min ||x|| = 1 Ax \cdot x$.

Theorem. (46)
Let G be a graph, $\Delta(G) = G$, λ an eigenvalue of G. Then $|\lambda| \le \Delta$. Moreover, if G is connected, then Δ is an eigenvalue $\iff$ G is Δ-regular, in this case, Δ has multiplicity 1 and eigenvector $(1, 1, ..., 1)^T$.

Proof. Let A be the adjacency matrix and x an eigenvector with eigenvalue λ. Let i be s..t $|x_i|$ is maximal. WLOG $x_i = 1$ and $\forall j$, $|x_j| \le 1$. Then

$$|\lambda| = |\lambda x_i| = |(Ax)_i)| = |\sum_{j=1}^n A_{ij}x_j| = |\sum_{j \in \Gamma(i)} x_j| \le \sum_{j \in \Gamma(i)} |x_j| \le d(i) \le \Delta$$

Assume now G is connected. We first show backward: if G is Δ-regular, then clearly $(1, 1, ..., 1)^T$ is eigenvector, with eigenvalue Δ.
To show forward, suppose Δ is an eigenvalue. Then taking $\lambda = \Delta$ in previous:

$$\Delta = (Ax)_i = \sum_{j \in \Gamma(i)} x_j$$

Hence $d(i) = \Delta$ and $\forall j \in \Gamma(i), x_j = 1$. Repeat: $\forall j \in \Gamma(i)$ we jave $d(j) = \Delta$ and $\forall k \in \Gamma(j), x_k = 1$. Continuing, as G connected, $\forall k$ we have $d(k) = \Delta$ and $x_k = 1$. So G is Δ-regular. $\qquad\square$

7.3   Strongly Regular Graphs

Let $k, b \ge 1$ and $a \ge 0$. A graph G is (k, a, b)-strongly-regular if G is k-regular and, for all $x, y \in G$ with $x \ne y$,
• $x \sim y \implies |\Gamma(x) \cap \Gamma(y)| = a$,
• $x \not\sim y \implies |\Gamma(x) \cap \Gamma(y)| = b$.

Note that a strongly regular graph is connected. Indeed, it has diamater ≤ 2.

A Moore graph is $(k, 0, 1$-s-r. Conversely, suppoer G is $(k, 0, 1)$-s-r with $k \ge 2$. Then $\Delta(G) = k$, G has diameter 2. Pick $x \in G$. Then

$$|G| = |\{x\}| + |\Gamma(x)| + |\Gamma(\Gamma(x)) \setminus \{x\}| = 1 + k + k^2 - k = k^2 + 1$$

so G is a Moore Graph.

Let G be (k, a, b)-s-r. Let A be the adjacency matrix of G. Let $|G| = n$. By theorem 46, k is an eigenvalue of multiplicity 1 with eigenvector $(1, 1, ..., 1)^T = v$. What about other eigenvalues? Let $\lambda \ne k$ be an eigenvalue with eigenvector x. Now

$$(A^2)_{ij} = \begin{cases} k & i = j \\ a & i \sim j \\ b & i \ne j, i \not\sim j \end{cases}$$

So $A^2 = kI + aA + b(J - I - A)$, where J is a matrix with every entry 1. Applying this to x, noting that $x \perp v$ (since we have an orthonormal eigenbasis) giving $Jx = 0$, we get $\lambda^2 x = kx + a\lambda x - bx - b\lambda x$. Also $x \ne 0$, so cancel x and we get

$$\lambda^2 + (b - a)\lambda + (b - k) = 0$$

So the remaining eigenvalues are $\lambda = \frac{(a-b)\pm\sqrt{(b-a)^2-4(b-k)}}{2}$ with multiplicities r, s for $+$ and $-$ respectively. Now A is diagonalizable so $r + s + 1 = n$ (1), and $\text{tr}(A) = 0$ so $\lambda r + \mu s + k = 0$ (2). Take $\lambda \times$ (1) - (2), we get $(\lambda - \mu)s = \lambda(n-1) + k$. We have $\lambda - \mu = \sqrt{(b-a)^2 - 4(b-k)}$, so $s = \frac{1}{2}\left[(n-1) + \frac{(a-b)(n-1)+2k}{\sqrt{(b-a)^2-4(b-k)}}\right]$, and so $r = \frac{1}{2}\left[(n-1) - \frac{(a-b)(n-1)+2k}{\sqrt{(b-a)^2-4(b-k)}}\right]$. Hence we have proved:

Theorem. (47, Rationality condition)

Let G be (k, a, b)-s-r. Then $\frac{1}{2}\left[(n-1) \pm \frac{(a-b)(n-1)+2k}{\sqrt{(b-a)^2-4(b-k)}}\right] \in \mathbb{Z}$ as they are multiplicites.

Example. Consider Moore graphs again: we have $a = 0$, $b = 1$, $n = k^2+1$. So we need $\frac{1}{2}\left[k^2 \pm \frac{2k-k^2}{\sqrt{4k-3}}\right] \in \mathbb{Z}$ as a neccesary condition for the corresponding Moore graph to exist. There are two possibilities: either $2k - k^2 = 0$, i.e. $k = 2$, or we need $k - 3 = u^2$ for some $u \in \mathbb{N}$ with $u|2k-k^2$. Then $k = \frac{u^2+3}{4}$ and $k^2 - 2k = ut$ for some $t \in \mathbb{Z}$. Hence $(\frac{u^2+3}{4})^2 - 2(\frac{u^2+3}{4}) = ut$. So $u^4 - 6u^2 + 9 - 8u^2 - 24 = 16ut$, so $u^4 - 2u^2 - 16ut - 15 = 0$. Hence $u|15$. So $u = 1, 3, 5, 15$. Hence $k = 1, 3, 7, 57$. Can we actually find these? $k = 1$ doesn't work as it's not diameter 2. We've previously seen that for $k = 2$ we have C_5, and for $k = 3$ we have the Petersen graph. $k = 7$? $n = 50$, and it is in fact possible to construct that graph. $k = 57$? n is too large, and this question is still open.

—end of course—