Geometry

March 14, 2017

Contents

1 Euclidean Geometry

1.1 Isometries

Let $(.,.)$ be the standard inner product (dot product) on the Euclidean space $\mathbb{R}^n$, i.e. for $x, y \in \mathbb{R}^n$ we have

$$(x, y) = x \cdot y = \sum_{i=1}^{n} x_i y_i$$

The Euclidean norm, $||x|| = \sqrt{(x, x)}$.
The Euclidean distance function, $d(x, y) = ||x - y||$.

We know that $(\mathbb{R}^n, d)$ is a metric space.

Definition. A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is an *isometry* of $\mathbb{R}^n$ if

$$d(f(P), f(Q)) = d(P, Q)$$

for all $P, Q \in \mathbb{R}^n$.

Isometries may be defined for any metric space.

Recall that a $n \times n$ matrix A is *orthogonal* if $A^T A = AA^T = I$.

For $x, y \in \mathbb{R}^n$,

$$\begin{aligned} (Ax, Ay) &= (Ax)^T (Ay) \\ &= x^T A^T Ay \\ &= (x, A^T Ay) \end{aligned}$$

So A is orthogonal iff $(Ax, Ay) = (x, y)$ for all $x, y \in \mathbb{R}^n$.

Now from the definition we see

$$(x, y) = \frac{1}{2}(||x + y||^2 - ||x||^2 - ||y||^2)$$

Thus A is orthogonal iff $||Ax|| = ||x||$ for all $x \in \mathbb{R}^n$.

If $f(x) = Ax + b$ for some $b \in \mathbb{R}^n$, then $d(f(x), f(y)) = ||A(x - y)||$.

So f is an isometry iff A is an orthogonal matrix.

Theorem. 1.1
Every isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ is of the form

$$f(x) = Ax + b$$

for some orthogonal A and $b \in \mathbb{R}^n$.

Proof. Let $e_1, ..., e_n$ be the standard basis. Put $f(0) = b, f(e_i) - b = a_i$ for $i = 1, ..., n$.

Then

$$\begin{aligned} ||a_0|| &= ||f(e_i) - f(0)|| \\ &= d(f(e_i), f(0)) \\ &= d(e_i, 0) \\ &= ||e_i|| \\ &= 1. \end{aligned}$$

for $i \neq j$,

$$\begin{aligned} (a_i, a_j) &= -\frac{1}{2}(||a_i - a_j||^2 - ||a_i||^2 - ||a_j||^2) \\ &= -\frac{1}{2}(||f(e_i) - f(e_j)||^2 - 2) \\ &= -\frac{1}{2}(||e_i - e_j||^2 - 2) \\ &= 0. \end{aligned}$$

Thus $\{a_i\}$ is an orthonormal basis.

So the matrix

$$A = (a_1 \quad a_2 \quad ... \quad a_n)$$

is orthogonal.

Now let $g(x) = Ax + b$. We just have to prove that $f = g$.

We know g is an isometry. Also, $g(x) = f(x)$ for $x = 0, e_1, ..., e_n$, and

$$g^{-1}(x) = A^{-1}(x - b) = A^T(x - b)$$

hence $h = g^{-1} \circ f$ is an isometry fixing $0, e_1, ..., e_n$.

We need to prove that $h = id$. Consider $x \in \mathbb{R}^n$. Write

$$x = \sum_{i=1}^{n} x_i e_i$$

and

$$y = h(x) = \sum_{i=1}^{n} y_i e_i$$

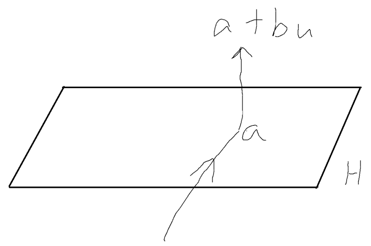
Then

$$\begin{aligned} &d(x, e_i)^2 = ||x||^2 + ||e_i|| - 2x_i, \\ &d(x, 0)^2 = ||x||^2, \\ &d(y, e_i)^2 = ||y||^2 + 1 - 2y_i, \\ &d(y, 0) = ||y||^2 \end{aligned}$$

h is an isomtery, $h(0) = 0$, $h(e_i) = e_i$, $h(x) = y$. So $||x||^2 = ||y||^2$. So $x_i = y_i$ for all i. So $h = id$. □

Let $\text{Isom}(\mathbb{R}^n)$ be the set of all isometries of $\mathbb{R}^n$. This is a group by composition (the group of rigid motions of $\mathbb{R}^n$).

Example. Consider Reflections in an affine hyperplane $H \subset \mathbb{R}^n$.



$$H = \{x \in \mathbb{R}^n : u \cdot x = c\}$$

where $||u|| = 1$, $c \in \mathbb{R}$ is a given constant.

Reflection in H:

$$R_H : x \to x - 2(x \cdot u - c)u$$

is an isometry (see example sheet).

Observe: if $x \in H$ then $R_H = x$.

If $a \in H$, $t \in \mathbb{R}$, then

$$\begin{aligned} R_H(a + tu) &= (a + tu) - 2((a + tu) \cdot u - c)u \\ &= (a + tu) - 2tu \\ &= a - tu \end{aligned}$$

That means R_H fixes precisely the points in H.

Conversely, suppose $S \in \text{Isom}(\mathbb{R}^n)$ and S fixes H.

Given $a \in H$, define translation by a: $T_a(x) = x + a$. Then set

$$R = T_{-a} S T_a \in \text{Isom}(\mathbb{R}^n)$$

R fixes $H' = T_{-a}(H)$ by inspection. Notice $0 \in H'$, so H' is a vector subspace of $\mathbb{R}^n$.

If $H = \{x \cdot u = c\}$, then $H' = \{x \cdot u = 0\}$.

Then, whenever $x \in H'$, we have

$$\begin{aligned} (Ru, x) &= (Ru, Rx) \\ &= (u, x) \\ &= 0 \end{aligned}$$

So $Ru \perp H'$, i.e. $Ru = \lambda u$ for some $\lambda \in \mathbb{R}$.

But $||Ru||^2 = 1$ as $||u||^2 = 1$, so $\lambda^2 = 1$, i.e. $\lambda = \pm 1$.

Since R fixes 0 ($0 \in H'$), R is a linear map by Theorem 1.1 and either $R = id_{R^n}$ or $R = R_{H'}$ (corresponding to the matrix $\text{Diag}(-1, 1, ..., 1)$).

So S is either $id_{\mathbb{R}^n}$ or $S = T_a R_{H'} T_{-a}$ is a reflection.

Checking S when

$$\lambda = -1 : x \to x - a \to (x - a) - 2((x - a) \cdot u)u \to x - 2(x \cdot u - c)u$$

noting $a \cdot u = c$. Thus $S = R_H$.

We find that R_H is the unique isometry of $\mathbb{R}^n$ which fixes H but is not identity.

It can be shown that every isometry of $\mathbb{R}^n$ is a composition of at most $n + 1$ reflections (example sheet 1).

From Theorem 1.1, the subgroup consisting of isometries fixing the origin is $\{f(x) = Ax : AA^T = I\}$ is naturally isomorphic to $O(n)$.

$A \in O(n) \implies (\det A)^2 = 1 \implies \det A = \pm 1$.

Definition. The *special orthogonal group*, $SO(n)$, consists of the matrices in $O(n)$ with determinant $+1$.

1.2 Orthogonal groups

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff a^2 + c^2 = 1, b^2 + d^2 = 1, ab + cd = 0 \iff A \in O(2). \quad (*)$$

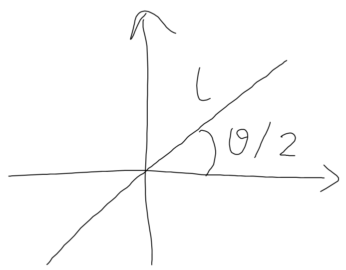
Set $a = \cos\theta$, $b = -\sin\varphi$, $c = \sin\theta$, $d = \cos\varphi$ for appropriate $0 \leq \theta, \varphi \leq 2\pi$. So (*) says $\tan\theta = \tan\varphi \in \mathbb{R} \cup \{\infty\}$. So $\theta = \varphi$ or $\theta = \varphi \pm \pi$. Respectively,

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is a rotation through θ about O. $\det A = 1$, so $A \in SO(2)$. The other possibility is

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

fixes a line l and must be a reflection in l (see graph below). We have $\det A = -1$.



Remark. *Orientation* of a vector space on equivalence class of bases.

• Let $v_1, ..., v_n$ and $v'_1, ..., v'_n$ and $A = (A_{ij})$ the respective matrix for change from $\{v_i\}$ to $\{v'_i\}$. Then the bases are "equivalent", i.e. have the same orientation iff $\det A > 0$.

We define an isometry $f(x) = Ax + b$ to be *orientation-preserving* if $\det A = 1$, *orientaiton-reversing* if $\det A = -1$.

Now we consider the group $O(3)$.

Consider first the case $\det A = 1$. Then

$$\det(A - I) = \det(A^T - I) = \det(A(A^T - I)) = \det(I - A)$$

But A has dimension 3. So $\det(A - I) = 0$. So $+1$ is an eigenvalue of A. So $\exists v_1 \in \mathbb{R}^3$ (WLOG let $||v_1|| = 1$) s.t. $Av_1 = v_1$.

Set $W = \langle v_1 \rangle^\perp$. Then

$$w \in W \implies (Aw, v_1) = (Aw, Av_1) = (w, v_1) = 0$$

So $A|_W$ is a rotation of 2-dimensional space W. Choose an orthonormal basis $\{v_2, v_3\}$ of W. Then w.r.t $\{v_1, v_2, v_3\}$, A becomes

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Now let $\det A = -1$. Then $-A$ has determinant 1, so is of the above form in some orthonormal basis. So A takes the form

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

with $\varphi = \theta + \pi$. This is a *rotated reflection* (pure reflection when $\phi = 0$).

1.3 Curves in $\mathbb{R}^n$

Definition. A *curve* Γ in $\mathbb{R}^n$ is a continuous function $\Gamma : [a, b] \to \mathbb{R}^n$.

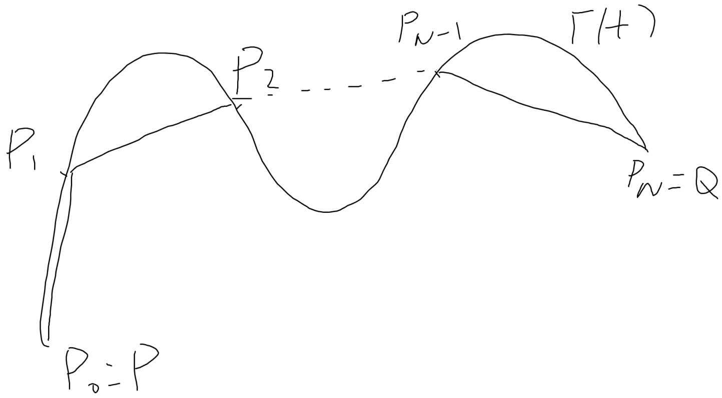
A dissection is $\mathcal{D} : a = t_0 < t_1 < ... < t_N = b$ of $[a, b]$.

Set $P_i = \Gamma(t_i) \in \mathbb{R}^n$, $S_{\mathcal{D}} = \sum_i ||\vec{P_i P_{i+1}}||$.

We define the *length* of Γ as

$$l = \sup_{\mathcal{D}} S_{\mathcal{D}}$$

if this exists (i.e. finite).



If $\mathcal{D} = (P_i = \Gamma(t_i))_{i=1}^N$ is a dissection of Γ and $\mathcal{D}'$ is a refinement (contain extra points) of $\mathcal{D}$, then $S_{\mathcal{D}} \leq S_{\mathcal{D}'}$ by triangle inequality.

Let $\text{Mesh}(\mathcal{D}) = \max_i(t_i - t_{i-1})$. Then, if the length l of Γ exists (i.e. finite), then we have

$$l = \lim_{\text{Mesh}(\mathcal{D} \to 0)} S_{\mathcal{D}}.$$

Note also $l = \min\{\tilde{l} : \tilde{l} \geq S_{\mathcal{D}} \forall \mathcal{D}\}$.

Proposition. 1.2
If Γ is continuously differentiable (C^1), then the length of Γ is

$$l = \int_a^b ||\Gamma'(t)||dt$$

Proof. Assume $n = 3$ to ease the notation. We have

$$\Gamma(t) = (f_1(t), f_2(t), f_3(t)).$$

Given $s \neq t$ in $[a, b]$, use MVT for each f_i, we get

$$\frac{f_i(t) - f_i(s)}{t - s} = f_i'(\xi_i)$$

for some $\xi_i \in (s, t)$.

f_i' is continuous on $[a, b]$. So f_i' is uniformly continuous. So $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ s.t. $|t - s| < \delta \implies |f_i'(xi_i) - f_i'(\xi)| < \varepsilon \ \forall \xi \in (s, t)$.

So

$$\begin{aligned} ||\frac{\Gamma(t) - \Gamma(s)}{t - s} - \Gamma'(\xi)|| &= ||(f_1'(\xi_1), f_2'(\xi_2), f_3'(\xi_3)) - (f_1'(\xi), f_2'(\xi), f_3'(\xi))|| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

i.e.

$$||\Gamma(t) - \Gamma(s) - (t-s)\Gamma'(\xi)|| < \varepsilon(t-s)$$

Now let $t = t_i, s = t_{i-1}, \xi = \frac{t_{i-1}+t_i}{2}$. So

$$(t_i - t_{i-1})||\Gamma'(\frac{t_{i-1}+t_i}{2})|| - \varepsilon(t_i - t_{i-1}) \leq ||\Gamma(t_i) - \Gamma(t_{i-1})|| \leq (t_i - t_{i-1})||\Gamma'(\frac{t_i+t_{i-1}}{2})|| + \varepsilon(t_i - t_{i-1})$$

So

$$\sum_i (t_i - t_{i-1})||\Gamma'(\frac{t_i+t_{i-1}}{2}|| - \varepsilon(b-a) < S_{\mathcal{D}} < \sum_i (t_i - t_{i-1})||\Gamma'(\frac{t_i+t_{i-1}}{2})|| + \varepsilon(b-a)$$

But $||\Gamma'(t)||$ is continuous, hence integrable. So

$$\sum_i (t_i - t_{i-1})||\Gamma'(\frac{t_i+t_{i-1}}{2})|| \to \int_a^b ||\Gamma'(t)||dt$$

as $\mathrm{Mesh}(\mathcal{D}) \to 0$.

Thus the length of Γ is

$$l = \lim_{\mathrm{Mesh}(\mathcal{D})\to 0} S_{\mathcal{D}} = \int_a^b ||\Gamma'(t)||dt.$$

□

2 Spherical Geometry

Denote $S = S^2 \subset \mathbb{R}^3$ the unit sphere in with centre origin.

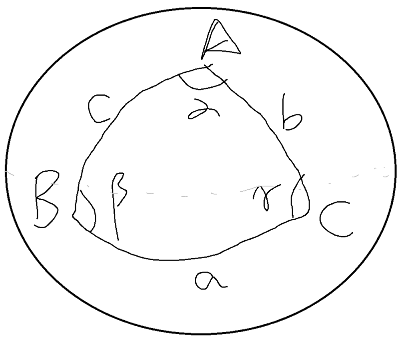
Definition. A *great circle* a.k.a (spherical) line in S^2, is $S^2 \cap$ a plane through the origin.

Given two distincts non-antipodal points $P, Q \in S^2$, there exists a unique line in S^2 through P, Q (as P, Q and the origin fix a plane).

Definition. For $P, Q \in S^2$, the distance $d(P, Q)$ is the length of the shorter of the two spherical line segments PQ along the great circle through P and Q. $d(P, Q) = \pi$ if P, Q are antipodal.

Note that $d(P, Q) =$ angle between $\mathbf{P} = \vec{OP}$ and $\mathbf{Q} = \vec{OQ} = \cos^{-1}(\mathbf{P} \cdot \mathbf{Q})$.

A *spherical triangle* ABC is defined like a Euclidean triangle, but with AB, BC, CA line segments in S^2 with lengths $< \pi$.



Notation. Write $\mathbf{A} = \vec{OA}$ and etc.
Set

$$\mathbf{n_1} = \frac{\mathbf{C} \times \mathbf{B}}{\sin a},$$
$$\mathbf{n_2} = \frac{\mathbf{A} \times \mathbf{C}}{\sin b},$$
$$\mathbf{n_3} = \frac{\mathbf{B} \times \mathbf{A}}{\sin c}.$$

These are unit normals to the planes OBC, OCA, OAB, pointing out of the solid $OABC$.

α, β, γ are the angle between planes defining respective sides of ABC.

Note $0 < \alpha, \beta, \gamma < \pi$. So (angle between them)$\widehat{n_2, n_3} = \pi - \alpha$, $\mathbf{n_2} \cdot \mathbf{n_3} = -\cos\alpha$. Similarly, $\mathbf{n_1} \cdot \mathbf{n_2} = -\cos\gamma$, $\mathbf{n_1} \cdot \mathbf{n_3} = -\cos\beta$.

Theorem. 2.1 (Spherical cosine rule)
For a spherical triangle, we have

$$\sin a \sin b \cos \gamma = \cos c - \cos a \cos b.$$

Proof. Use $(\mathbf{C} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{C}) - (\mathbf{C} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{A})$ and

$$\sum_k \varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

from vector calculus. We know $|\mathbf{C}| = 1$. So

$$RHS = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{C}) - (\mathbf{B} \cdot \mathbf{A})$$

So

$$-\cos \gamma = \mathbf{n}_1 \cdot \mathbf{n}_2 = \frac{\mathbf{C} \times \mathbf{B}}{\sin a} \cdot \frac{\mathbf{A} \times \mathbf{C}}{\sin b} = \frac{(\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{C}) - (\mathbf{A} \cdot \mathbf{B})}{\sin a \sin b} = \frac{\cos b \cos a - \cos c}{\sin a \sin b}$$

which is equivalent to what is required. □

Corollary. 2.2 (Pythagoras for S^2)
If $\gamma = \frac{\pi}{2}$, then $\cos c = \cos a \cdot \cos b$.

Theorem. 2.3 (Spherical sine rule)
For a spherical triangle, we have

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}$$

Proof. Use

$$(\mathbf{A} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{B}) = (\mathbf{C} \cdot (\mathbf{B} \times \mathbf{A}))\mathbf{C}$$

from vector calculus. Recall $\widehat{n_1, n_2} = \pi - \gamma$. We have

$$LHS = -(\mathbf{n}_1 \times \mathbf{n}_2) \sin a \sin b$$

So $\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{C} \sin \gamma$, as from RHS we see that this is a multiple of $\mathbf{C}$. So

$$\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \sin a \sin b \sin \gamma = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \sin b \sin c \sin \alpha$$

Multiply by $\frac{1}{\sin \alpha \sin \beta \sin \gamma}$ we get

$$\frac{\sin c}{\sin \gamma} = \frac{\sin b}{\sin \beta} = \frac{\sin a}{\sin \alpha}$$

□

We have seen cosine and sine rules for spherical triangles. There is a second cosine rule (Sheet 1 Q15).

Remark. Recall for small a, b, c, $\sin a = a + O(a^3)$, $\cos a = 1 - \frac{a^2}{2} + O(a^4)$. We get the Euclidean versions in the limit $a, b, c \to 0$.

e.g. in Theorem 2.1,

$$\begin{aligned} ab\cos\gamma &= 1 - \frac{c^2}{2} - \left(1 - \frac{a^2}{2}\right)\left(1 - \frac{b^2}{2}\right) + O(||(a,b,c)||^3) \\ \implies c^2 + 2ab\cos\gamma &= a^2 + b^2 + O(||(a,b,c)||^3). \end{aligned}$$

If $\gamma = \pi$, then C is in the line segment AB. So $c = a + b$. Otherwise from Theorem 2.1, $\cos c > \cos a \cos b - \sin a \sin b = \cos(a + b)$, so $c < a + b$. Also $c < \pi, a + b < 2\pi$.

Corollary. (Triangle inequality)
$\forall P, Q, R \in S^2$, we have $d(P, Q) + d(Q, R) \geq d(P, R)$ (spherical distance), with equality only if Q is in the line segment PR of the shorter length.

Proof. The only case not covered by the previous discussion is when $d(P, R) = \pi$, i.e. P, R antipodal. Then R is in the line PQ. So $d(P, R) = d(P, Q) + d(Q, R)$. □

So we find that (S^2, d) is a *metric space.*

Proposition. 2.5
Given a curve Γ on S^2 from P to Q with $l = length(\Gamma)$, we have

$$l \geq d(P, Q)$$

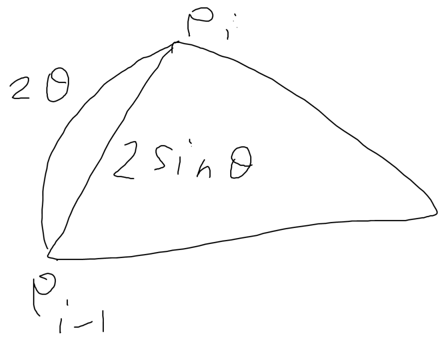
Moreover, if $l = d(P, Q)$ then Γ is a spherical line segment.

Proof. $\Gamma : [0, 1] \to S^2$. $length(\Gamma) = l \implies \forall$ dissection $\mathcal{D}$ of $[0, 1]$: $0 = t_0 < t_1 < ... < t_N = 1$, $p_i = \Gamma(t_i)$,

$$\tilde{\mathcal{S}_\mathcal{D}} := \sum_{i=1}^N d(p_{i-1}, p_i) > \mathcal{S}_\mathcal{D} = \sum_{i=1}^N |p_{i-}\vec{1}p_i|$$

where RHS is $\mathbb{R}^3$ distance.

Using the fact $\sin\theta < \theta \ \forall\theta > 0$,



Now suppose $l < d(P,Q)$. Then we can choose $\varepsilon > 0$ s.t. $(1+\varepsilon)l < d(P,Q)$. Now since $\frac{\sin\theta}{\theta} \to 1$ as $\theta \to 0$, $2\theta \le (1+\varepsilon)2\sin\theta$ for small $\theta > 0$.

Γ is uniformly continuous on $[0,1]$. So we can choose a refined $\mathcal{D}$ with $d(p_{i-1},p_i) \le (1+\varepsilon)|p_{i-1}\vec{p}_i|$. So

$$\tilde{\mathcal{S}}_{\mathcal{D}} \le (1+\varepsilon)\mathcal{S}_{\mathcal{D}} \le (1+\varepsilon)l < d(P,Q)$$

But $\tilde{\mathcal{S}_{\mathcal{D}}} \ge d(P,Q)$ by triangle inequality (applied many times). Contradiction. So $l \ge d(P,Q)$.

Suppose now $l = d(P,Q)$ for some $\Gamma : [0,1] \to S$. Then $\forall t \in [0,1]$,

$$\begin{aligned} d(P,Q) = l &= length\Gamma|_{[0,t]} + length\Gamma|_{[t,1]} \\ &\ge d(P,\Gamma(t)) + d(\Gamma(t),Q) \end{aligned}$$

So $d(P,Q) = d(P,\Gamma(t)) + d(\Gamma(t,Q)$ $\forall t$. So $\Gamma(t)$ is in the shorter spherical line segment PQ. □

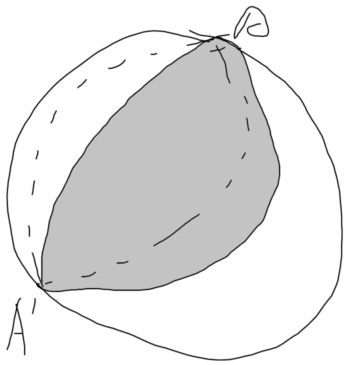
Sheet 1 Q4 is the Euclidean version of this discussion.

Remark. If Γ is a curve in S^2 of minimal length from P to Q, then Γ is a spherical line segment. Further, from the proof of proposition 2.5, $length(\Gamma|_{[0,t]}) = d(P,\Gamma(t))$ $\forall t \in [0,1]$. So the parameterisation of Γ is *monotonic*, i.e. the distance increases as t increases.

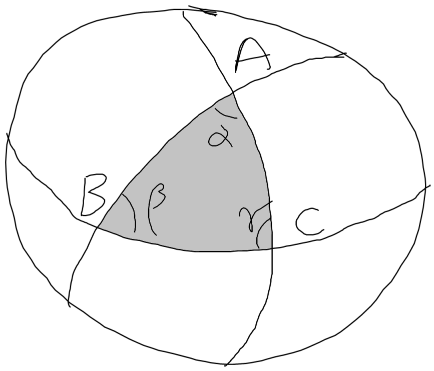
Proposition. 2.6 (Gauss-Bonnet theorem for S^2)
If Δ is a spherical triangle with angles α, β, γ, then

$$area(\Delta) = (\alpha + \beta + \gamma) - \pi.$$

Proof. A *double lune* with angle $0 < \alpha < \pi$ is two regions on S cut out by 2 planes through antipodal points, say A and A', where α is the angle between the plane.



The area of double lune is 4α (noting it is proportional to α, and $area(S^2) = 4\pi$).



$\Delta = ABC$ is the intersection of 3 single lunes. So Δ and its antipodal Δ' is a subset of each of 3 double lunes with angles α, β, γ.

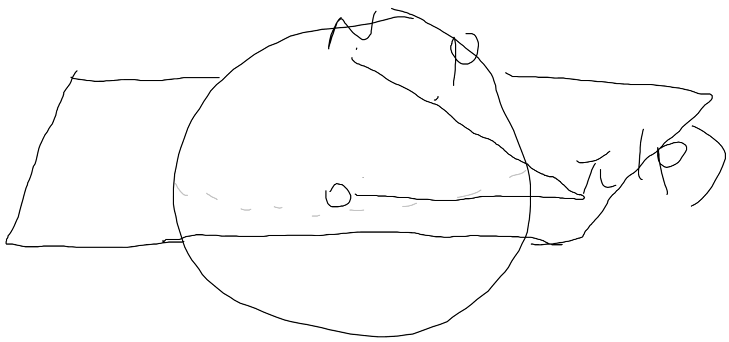
Any other $P \notin \Delta \cup \Delta'$ is in only one double lune.

Thus $4(\alpha + \beta + \gamma) = 4\pi + 2 \cdot (2\Delta)$ which gives the desired result. □

Remark. (i) On S, we have $\alpha + \beta + \gamma > \pi$ ($\to \pi$ as $a, b, c \to 0$).
(ii) For convex n-gon, $area(M) = \sum_{i=1}^{n} \alpha_i - (n-2)\pi$ (cut into triangles).

2.1 Möbius geometry

Consider $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ with coordinates $\zeta = x + iy$.
The stereographic projection $\pi : S^2 \to \mathbb{C}_\infty$:



is $\pi(P) = (NP) \cap \{z = 0\} \cong \mathbb{C} \cong \mathbb{R}^2$, $\pi(N) = \infty$ where $N = (0, 0, 1)$.

By Euclidean geometry we can get

$$\pi(x, y, z) = \frac{x + iy}{1 - z}$$

Lemma. 2.7
If π' is the stereographic projection from $(0, 0, -1)$ (South pole), then

$$\pi'(P) = \frac{1}{\overline{\pi(P)}}$$

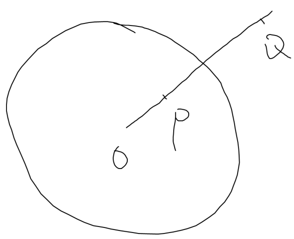
$\forall P \in S^2$.

Proof. Let $P = (x, y, z)$. Then $\pi(P) = \frac{x+iy}{1-z}$, $\pi'(P) = \frac{x+iy}{1+z}$. So

$$\overline{\pi(P)} \cdot \pi'(P) = \frac{x^2 + y^2}{1 - z^2} = 1$$

□

Note: $\pi' \circ \pi^{-1} : \mathbb{C} \to \mathbb{C}$ takes ζ to $\frac{1}{\bar{\zeta}}$, the *inversion in the unit circle* $\{x^2 + y^2 = 1\} = \{|\zeta| = 1\}$.



If $P = (x, y, z) \in S^2$, $-P = (-x, -y, -z)$, then $\pi(P) = \frac{x+iy}{1-z}$, $\pi(-P) = \frac{-x-iy}{1+z}$. So

$$\pi(P) \cdot \overline{\pi(-P)} = \frac{-(x^2+y^2)}{1-z^2} = -1.$$

So $\pi(-P) = -\frac{1}{\bar{\zeta}}$.

Möbius transformations act on $\mathbb{C}_\infty$ and form a group G by composition. Any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C})$ defines a Möbius map

$$\zeta \to \frac{a\zeta + b}{c\zeta + d}.$$

For all $\lambda \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, λA defines the same Möbius transformation.

Conversely, if A_1, A_2 give the same transformation, then $\exists \lambda \neq 0$ s.t. $A_1 = \lambda A_2$.

So $G \cong PGL(2, \mathbb{C}) = GL(2, \mathbb{C})/\mathbb{C}^*$. i.e. $\mathbb{C}^* \cong \{\lambda I : \lambda \in \mathbb{C}^*\}$ is a normal subgroup.

It suffices to consider $\det A = 1$. If $\det \tilde{A} = 1$, $A = \lambda \tilde{A}$, then $1 = \det(\lambda \tilde{A}) = \lambda^2 \det A = \lambda^2$, i.e. $\lambda = \pm 1$.

So $G \cong PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\pm I$ (group homomorphism $SL(2, \mathbb{C}) \to G$.

On S^2 we have rotations $SO(3)$ acting as isometries (see Q5 ES 1).

Theorem. 2.8
Via the stereographic projection π, every rotation of S^2 induces a Möbius map defined by a matrix in the subgroup $SU(2) \subset SL(2, \mathbb{C})$ (the Special Unitary group of degree n is the group of $n \times n$ orthogonal matrix with determinant 1). In the case $n = 2$, we have

$$SU(2) = \left\{ \begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\}$$

(Incidentally, $SU(2) \leftrightarrow S^3 \subset \mathbb{R}^4$).

Proof. (1) rotations $r(z, \theta)$ about the $z-$axis $\mathbb{R}(0, 0, 1)$ through angle θ. The corresponding Möbius map is $\zeta \to e^{i\theta}\zeta$, i.e. a rotation of the complex plane, with matrix

$$\begin{pmatrix} e^{\frac{i\theta}{2}} & 0 \\ 0 & e^{-\frac{i\theta}{2}} \end{pmatrix} \in SU(2).$$

(2) rotation $r(y, \frac{\pi}{2})$ is

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ -x \end{pmatrix}$$

Which is rotation about $y-$axis through $\pm i$, sending $-1 \to \infty$, $1 \to 0$, $i \to i$. There is *only one* such Möbius map

$$\zeta' = \frac{\zeta - 1}{\zeta + 1}$$

checking, this Möbius map gives $r(y, \frac{\pi}{2})$: $\zeta = \frac{x+iy}{1-z}$. So

$$\begin{aligned}\frac{\zeta-1}{\zeta+1} &= \frac{x+iy-1+z}{x+iy+1-z} = \frac{x-1+z+iy}{x+1-(z-iy)} = \frac{(z+iy)(x-1+z+iy)}{(x+1)(z+iy)+(x^2-1)} \\ &= \frac{(z+iy)(x-1+z+iy)}{(x+1)(z+iy+x-1)} = \frac{z+iy}{1+x} = \zeta'\end{aligned}$$

$r(y, \frac{\pi}{2})$ corresponds to Möbius map with

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in SU(2).$$

(3) $SO(3)$ is generated by $r(y, \frac{\pi}{2})$ and $r, (z, \theta)$ for $0 \le \theta < 2\pi$.

Observe $r(x, \varphi) = r(y, \frac{\pi}{2}) r(z, \varphi) r(y, -\frac{\pi}{2})$ (we can see that by considering the image of e_x under this map).

Also, $\forall \mathbf{v} \in S^2$ which is some unit vector, we can find φ, ψ s.t. $g = r(z, \psi) r(x, \varphi)$: $\mathbf{v} \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$r(x, \varphi)$ rotates $\mathbf{v}$ into the $(x, y)-$plane. Then for any given rotation we can write

$$r(\mathbf{v}, \theta) = g^{-1} r(x, \theta) g$$

(4) Thus, via π, any rotation of S^2 correspond to a composition of Möbius maps of $\mathbb{C}_\infty$ with matrices in $SU(2)$. □

This theorem gives a group homomorphism via π of $SO(3)$ and $PSU(2) = SU(2)/\pm I$. This is injective. In fact it is also surjective, so this is an isomorphism.

Theorem. 2.9
The group $SO(3)$ of rotations of S^2 corresponds precisely with the subgroup $PSU(2) = SU(2)/\pm I$ of Möbius transformations acting on $\mathbb{C}_\infty$.

Proof. Let $g \in PSU(2) \subset G$. Then

$$g(z) = \frac{az - b}{\bar{b}z + \bar{a}}$$

Suppose first $g(0) = 0$, so $b = 0$, $a\bar{a} = 1$, $a = e^{\frac{i\theta}{2}}$ for some real θ. Then g corresponds to $r(z, \theta)$, i.e rotation about $z-$axis through θ (notation of the proof of Theorem 2.8).

In general, $g(0) = w \in \mathbb{C}_\infty$. Let $Q \in S^2$, $\pi(Q) = w$. Choose $A \in SO(3)$ with $A(Q) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$. Let $\alpha \in PSU(2)$ the corresponding Möbius map (exists by Theorem 2.8). Then $\alpha(w) = 0$, $\alpha \circ g$ fixes 0. Hence $\alpha \circ g$ corresponds to $B = r(z, \tilde{\theta})$. Thus g corresponds to $A^{-1}B$. □

We've now shown that there is a 2−to−1 map $SU(2) \to PSU(2) \cong SO(3)$ and a group homomorphism $SU(2) \cong S^3$.

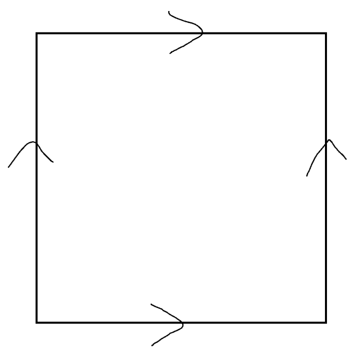
3 Triangulations and the Euler number

First, let's introduce one more 'geometry' - the *locally Euclidean torus.*

Definition. The *torus* T is the set $\mathbb{R}^2/\mathbb{Z}^2$ of equivalence classes of $(x, y) \in \mathbb{R}^2$ with equivalence relation

$$(x_1, y_1) \sim (x_2, y_2) \iff \begin{cases} x_1 - x_2 \in \mathbb{Z} \\ y_1 - y_2 \in \mathbb{Z} \end{cases}$$

Thus a point in T represented by (x, y) is a coset $(x, y) + \mathbb{Z}^2$ of the subgroup $\mathbb{Z}^2$ of the additive group $\mathbb{R}^2$.



For any closed square $Q \subset \mathbb{R}^2$ with side length 1, define the *distance* d, for $P_1, P_2 \in T$ to be

$$d(P_1, P_2) = \min\left\{|\mathbf{v}_1 - \mathbf{v}_2| \mid \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2, \mathbf{v}_i + \mathbb{Z}^2 = P_i \ \forall i\right\}.$$

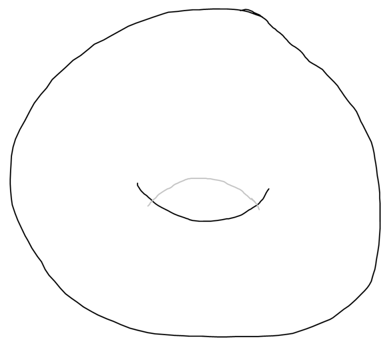
It's easy to check that (T, d) is a *metric space.*

Let Q° denote the interior of Q. We have a natural map $f : Q^\circ \to T$ a natural bijection onto open $U \subset T$.

If $P \in Q^\circ$, then f restricted to a small open disc about P is an isometry. So $f : Q^\circ \to U$ is a homomorphism.

d is said to be a *locally Euclidean distance function* (for Euclidean metric).

Remark. T may also be 'embedded' in $\mathbb{R}^3$.



The distance function we set by considering curves in $T \subset \mathbb{R}^3$ is *not* the same.

Definition. A *topological triangle* on X (here we usually consider X being either S^2 or T) is the image $R \subset X$ of closed Euclidean triangle $\Delta \subset \mathbb{R}^2$ under a homomorphism $\Delta \to R$.

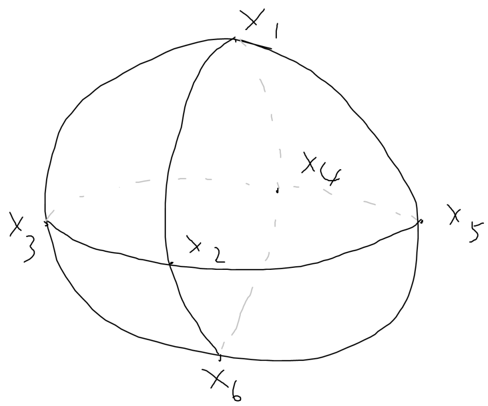
Example. A spherical triangle is a topological triangle (use a radial projection to a plane in $\mathbb{R}^3$ from O).

Definition. A (topological) *triangulation* τ of X is a finite collection of topological triangles on X s.t.
• $\forall$ two triangles are either disjoint or meet in exactly one edge or meet in exactly one vertex;
• each edge belongs to exactly two triangles.

Definition. The *Euler number* $e = e(X, \tau)$ is $e = F - E + V$ where F is the number of triangles, E is the number of edges, and V is the number of vertices.

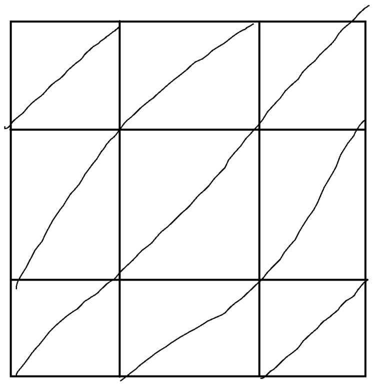
A fact from algebraic topology: e is independent of the choice of τ, so in fact $e = e(X)$.

Example. Consider $X = S^2$.



We have $F = 8$, $E = 13$, $V = 6$. So $e = 2$.

Example. Consider $X = T$ (imagine the diagonals are straight lines).



We have $F = 18$, $E = 27$, $V = 9$. So $e = 0$.

Note that in both cases we used *geodesic triangles*, i.e. edges are spherical or Euclidean lines of S^2 or T respectively.

Remark. Take a look again at the definition of a *triangulation*. We impose $X = \bigcup_{i=1}^{F} \Delta_i$ (can be deduced from other conditions – exercise).

Proposition. 3.1
For every geodesic triangles of S^2 or T, we have e being 2 or 0 respectively.

Proof. Denote 'faces' of triangles $\Delta_1, ..., \Delta_F$, and $\tau_i = \alpha_i + \beta_i + \gamma_i$, $i = 1, ..., F$, where $\alpha_i, \beta_i, \gamma_i$ are interior angles of the respective triangles. Then

$$\sum \tau_i = 2\pi V.$$

Also, $3F = 2E$ since every face has 3 edges and every edge is shared by 2 faces. So $F = 2E - 2F$.

In the case of S^2, by Gauss-Bonnet for S^2 (Proposition 2.6), area $\Delta_i = \tau_i - \pi$. So

$$\begin{aligned} 4\pi &= \sum_{i=1}^{F} \Delta_i = \sum_{i=1}^{F} (\tau_i - \pi) = 2\pi V - \pi F \\ &= 2\pi V - 2\pi E + 2\pi F \\ &= 2\pi e \end{aligned}$$

So $e = 2$.

In the case of torus T, we have $\tau_i = \pi \ \forall i$ as T is locally Euclidean. So

$$2\pi V = \sum_{i=1}^{F} \tau_i = \pi F$$

So $2V = F = 2E - 2F$. So $V - E + F = 0$. $\square$

Remark. We may use topological polygonal decomposition (rather than topological triangles), and proposition 3.1 will still hold. Then considering S^2, obtain Euler's formula

$$V - E + F = 2.$$

4 Hyperbolic Geometry

• Revision of derivatives and the chain rule: let $U \subset \mathbb{R}^n$ be open, $f = (f_1, ..., f_n) : U \to \mathbb{R}^m$ is smooth (C^∞) if each f_i has continuous partial derivatives of every order. This certainly implies differentiability (1st order partial derivatives are continuous).

The derivative of f at $a \in U$ is a linear map $df_u : \mathbb{R}^n \to \mathbb{R}^m$ (i.e. $DF|_a$ in Analysis II), so that

$$\frac{||f(a+h) - f(a) - df_a \cdot h||}{||h||} \to 0$$

as $h \to 0$ in $\mathbb{R}^n$.

If $m = 1$, then df_n is expressed as $\left(\frac{\partial f}{\partial x_1}(a), ..., \frac{\partial f}{\partial x_i}(a)\right)$ via

$$(h_1, ..., h_n) \to \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) h_i$$

For general m, we may use the *Jacobi matrix*

$$J(f)_a = \left(\frac{\partial f_i}{\partial x_j}(a)\right)$$

and $\mathbf{h} \to J(f)_a \mathbf{h}$.

Example. Holomorphic (analytic) functions of complex variable $f : U \subset \mathbb{C} \to \mathbb{C}$. $f'(z)$ is defined by

$$\frac{|f(z+w) - f(z) - f'(z)w|}{|w|} \to 0$$

as $w \to 0$. Let $f'(z) = a + ib$, $w = h_1 + ih_2$. Then

$$f'(z)w = (ah_1 - bh_2) + i(ah_2 + bh_1)$$

now $R^2 \cong \mathbb{C}$, $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ then $df_z : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Let $U \subset \mathbb{R}^n$, $v \subset \mathbb{R}^p$ be open, $f : U \to \mathbb{R}^m$, $g : V \to U$ be smooth functions. Then

$$f \circ g : V \to \mathbb{R}$$

has derivative

$$d(f \circ g)_p = (df)_{g(p)} \circ (dg)_p$$

for $p \in V$. Or, using the Jacobi matrices,

$$J(f \circ g)_p = J(f)_{g(p)} J(g)_p$$

by matrix multiplication.

4.1 Riemannian metrics (on open sets of $\mathbb{R}^2$)

We use coordinates $(u, v) \in \mathbb{R}^2$, let $V \subset \mathbb{R}^2$ be open. A Riemannian matrix is defined by giving C^∞ functions $E, F, G : V \to \mathbb{R}$ s.t.

$$\begin{pmatrix} E(p) & F(p) \\ F(p) & G(p) \end{pmatrix}$$

is a positive-definite matrix for every $p \in V$.

Thus $\forall p \in V$, the 2×2 matrix defines an inner product in $\mathbb{R}^2$(c.f. Linear Algebra), i.e.

$$\begin{aligned} \langle e_1, e_1 \rangle_p &= E(p), \\ \langle e_2, e_2 \rangle_p &= G(p), \\ \langle e_1, e_2 \rangle_p &= F(p). \end{aligned}$$

e.g. $E = G = 1$, $F = 0$ gives the standard Euclidean inner product.

Notation. We introduce the notation $Edu^2 + 2Fdudv + Gdv^2$, where $u : V \to \mathbb{R}$, $v : V \to \mathbb{R}$ the coordinates are C^∞ functions.

$du_p, dv_p : \mathbb{R}^2 \to \mathbb{R}$ have derivatives $(h_1, h_2) \to h_1, (h_1, h_2) \to h_2$.

Thus $du = du_p$, $dv = dv_p$ are elements of the dual space $(\mathbb{R}^2)^*$. Moreover they are LI. So they form a basis of $(\mathbb{R}^2)^*$, which is the dual basis to the standard basis of $\mathbb{R}^2$.

Thus $du^2, dudv, dv^2$ are bilinear forms on $\mathbb{R}^2$, with

$$\begin{aligned} du^2(h, k) &= du(h)du(k), \\ dudv(h, k) &= \frac{1}{2}(du(h)dv(k) + du(k)dv(h), \\ dv^2(h, k) &= dv(h)dv(k) \end{aligned}$$

corresponding to the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and so

$$Edu^2 + 2Fdudv + Gdv^2$$

is of the form

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

Definition. The *length* of a smooth curve $\gamma = (\gamma_1(t), \gamma_2(t)) : [0, 1] \to V \subset \mathbb{R}^2$ is

$$\int_0^1 \left(E\dot{\gamma}_1^2 + 2F\dot{\gamma}_1\dot{\gamma}_2 + G\dot{\gamma}_2^2 \right)^{1/2} dt$$

where the dot represents derivatievs with respect to t. Note that the integrand is just $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}$ (c.f. proposition 1.2).

The *area* of a region $W \subset V$ is defined as

$$\int_W (EG - F^2)^{1/2} du dv$$

which is the Gram determinant.

Example. Consider $V = \mathbb{R}^2$ with Riemmanian metric

$$\frac{4(du^2 + dv^2)}{(1 + u^2 + v^2)^2}$$

we shall see that via stereographic projection, $\pi : S^2 \setminus \{N\} \to \mathbb{R}^2_{u,v}$.

Recap on the Riemannian metrics. Suppose we have an open $V \subset \mathbb{R}^2$. We may think of $\mathbb{R}^2$ as an affine space A^2, or a vector space $\mathbb{R}^2$. It's easy to have identification $A^2 \cong \mathbb{R}^2$ (need to choose where to map the $\mathbf{0} \in \mathbb{R}^2$). We can attach a copy of $\mathbb{R}^2$ at $P \in A^2$.

Now $P \in S^2 \setminus \{N\}$, $P \neq N$. The tangent plane to S^2 at P is

$$\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot \overrightarrow{OP} = 0\}$$

$\mathbf{x} = \overrightarrow{OX} - \overrightarrow{OP}$. Consider $\pi(P) = (u, v) \in \mathbb{R}^2$ where π is the stereographic projection.

Example. (see sheet 3)
For all $x_1, x_2 \perp \overrightarrow{OP}$, $\mathbf{x}_1 \cdot \mathbf{x}_2 = \langle d\pi|_P(\mathbf{x}_1), d\pi|_P(\mathbf{x}_2)\rangle_{\pi(P)}$.

This formula defines an inner product $\langle \cdot, \cdot \rangle_{\pi(P)}$ on a 'copy of $\mathbb{R}^2$' at $\pi(P)$. Thus we induced an instance of Riemannian metric on $V = \mathbb{R}^2$ using $d\pi_P$ for $P \in S^2 \setminus \{N\}$.

Definition. Let $V, \tilde{V} \subset \mathbb{R}^2$ be open and endowed with Riemannian metrics. Denote $\langle \cdot, \cdot \rangle_P$, $O \in V$ and $\langle \cdot, \cdot \rangle^{\sim}_Q$, $Q \in \tilde{V}$ the respective inner products.

A *diffeomorphism* $\varphi : V \to \tilde{V}$ is called an *isometry* iff for all $P \in V$, $Q = \varphi(p)$ we have

$$\langle \mathbf{x}, \mathbf{y} \rangle_P = \langle d\varphi_P(\mathbf{x}), d\varphi_P(\mathbf{y}) \rangle^{\sim}_{\varphi(P)=Q}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$.

If $\gamma : [0, 1] \to V$ be a C^1 curve, then $\tilde{\gamma} = \varphi \circ \gamma : [0, 1] \to \tilde{V}$ is also a C^1 curve. Let $P = \gamma(t)$, so $\varphi(P) = \tilde{\gamma}(t)$. We have

$$\langle \tilde{\gamma}'(t), \tilde{\gamma}'(t) \rangle_{\tilde{\gamma}(t)} = \langle d\varphi_P(\gamma'(t)), d\varphi_P(\gamma'(t)) \rangle_{\varphi(P)}$$

by chain rule. If φ is an isometry then the above is equal to

$$\langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)}$$

Then (by integrating)

$$length(\tilde{\gamma}) = length(\gamma) = \int_0^1 \langle \gamma'(t), \gamma'(t) \rangle^{1/2}_{\gamma(t)} \, dt.$$

So isometries preserve lengths of curves, and so distances.

4.2 Two models for the hyperbolic plane

Definition. The *Poincare's disc model* for the hyperbolic plane is given by $D \subset \mathbb{C} \cong \mathbb{R}^2$, $D = \{|\zeta| < 1\}$ and a Riemannian metric

$$\frac{4(du^2 + dv^2)}{(1 - u^2 - v^2)^2} = \frac{4|d\zeta|^2}{(1 - |\zeta|^2)^2} \qquad (*)$$

where $\zeta = u + iv$, $d\zeta = du + idv$ (e.g. $d\zeta : \mathbb{C} \to \mathbb{C}$ linear map). Thus element of the dual complex vector space(??). $|d\zeta|^2 = du^2 + dv^2$.

(*) is a scaling of the Euclidean metric du^2+dv^2 by a factor depending on the polar radius $r = |\zeta|$: distances are scaled by $\frac{2}{1-r^2}$ and areas by $\frac{4}{(1-r^2)^2} = \sqrt{EG - F^2}$.

The upper half plane is $H = \{z \in \mathbb{C} : \Im(z) > 0\}$. D bijects to H via Mӧbius transformation $\zeta \in D \to \frac{i(1+\zeta)}{1-\zeta} \in H$.

We fix notation $z \in H$, $z = x + iy$, $z = \frac{1(i+\zeta)}{1-\zeta}$, $\zeta \in D$, $\zeta = u + iv$, $\zeta = \frac{z-i}{z+i}$.

We shall prove this induces a Riemann metric on H, so that $\zeta \to z$ as the above Mӧbius map is an isometry $D \to H$.

The Euclidean product on $\mathbb{C}(\cong \mathbb{R}^2)$ is $\langle w_1, w_2\rangle = \Re(w_1\bar{w}_2) = \frac{w_1\bar{z}_2 + \bar{w}_1 w_2}{2}$.

So if $\langle\cdot,\cdot\rangle$ is Euclidean at ζ, then at z s.t. $\zeta = \frac{z-i}{z+i}$ we require

$$\langle w_1, w_2\rangle_z = \left\langle \frac{d\zeta}{dz} w_1, \frac{d\zeta}{dz} w_2 \right\rangle_{Eud} = \left|\frac{d\zeta}{dz}\right|^2 \Re(w_1\bar{w}_2)$$

i.e. on H, we obtain a Riemannian metric

$$\left|\frac{d\zeta}{dz}\right|^2 (dx^2 + dy^2) = |dz^2|$$

We compute

$$\frac{d\zeta}{dz} = \frac{1}{z+i} - \frac{z-i}{(z+i)^2} = \frac{2i}{(z+i)^2},$$

$$1 - |\zeta|^2 = 1 - \frac{|z-i|^2}{|z+i|^2}$$

so

$$\frac{1}{1 - |\zeta|^2} = \frac{|z+i|^2}{|z+i|^2 - |z-i|^2} = \frac{|z+i|^2}{4\Im z}$$

Putting everything together, the metric on H corresponding $\frac{4|d\zeta|^2}{(1-|\zeta|^2)^2}$ is

$$4 \cdot \frac{4}{|z+i|^4} \cdot \left(\frac{|z+i^2}{4\Im z}\right)^2 \cdot |dz|^2 = \frac{|dz|^2}{(\Im z)^2} = \frac{dx^2 + dy^2}{y}$$

Note that on H we got a scaling of Euclidean matric: distances scaled by $1/y$ and areas scaled by $1/y^2$.

Definition. The *upper half-plane* model for the hyperbolic plane is H with metric
$$\frac{dx^2+dy^2}{y^2}$$

Consider $PSL(2,\mathbb{R}) = \left\{ z \to \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{R}, ad-bc=1 \right\}$, the subgroup of Mӧbius transformations sending $\mathbb{R}\cup\{\infty\} \to \mathbb{R}\cup\{\infty\}$ and $H \to H$.

Proposition. 4.1
The elements of $PSL(2,\mathbb{R})$ are *isometries* of H and thus preserve lengths of curves.

Proof. Easy to check that $PSL(2,\mathbb{R})$ is generated by:
$z \to z+a$, $a \in \mathbb{R}$;
$z \to az$, $a \in R^+$;
$z \to -1/z$.

It suffices to show that every of these three maps preserves the Riemannian metric
$$\frac{|dz|^2}{(\Im z)^2} = \frac{dx^2+dy^2}{y^2}$$
□

The first two are clear. We check the third one $f(z) = -1/z$:
$w \to f'(z)w$, $f'(z) = 1/z^2$, so
$$d\left(\frac{-1}{z}\right) = \frac{dz}{z^2},$$
$$\left|d\left(\frac{-1}{z}\right)\right|^2 = \frac{|dz|^2}{|z|^4},$$
$$\Im\left(\frac{-1}{z}\right) = \frac{-1}{|z|^2}\Im\bar{z} = \frac{\Im z}{|z|^2}$$
Thus
$$\frac{|d(-1/z)|^2}{|\Im(-1/z)|^2} = \frac{1/|z|^4|dz|^2}{(\Im z)^2/|z|^4} = \frac{|dz|^2}{(\Im z)^2}$$

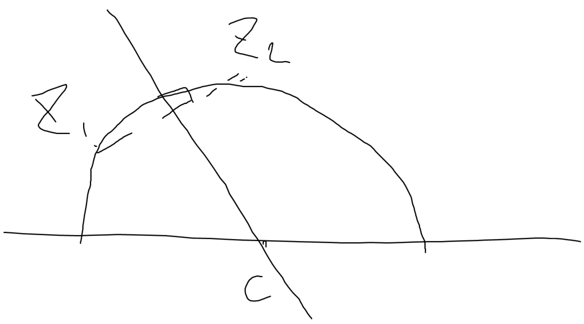
Remark. Each $z \to az+b$ for $a,b \in \mathbb{R}$, $a>0$ in $PSL(2,\mathbb{R})$ Hence $PSL(2,\mathbb{R})$ acts *transitively* on H.

Each Mӧbius transformation preserves the set of circles and straight lines in $\mathbb{C}$. If $L = i\mathbb{R}$, $g \in PSL(2,\mathbb{R})$, then $g(L)$ is either a circle centred at a point in $\mathbb{R}$ or straight line perpendicular to $\mathbb{R}$.

Put $L^+ = \{it : t > 0\}$. Then $g(L^+)$ is either a semicircle with ends in $\mathbb{R}$ or vertical half line starting at a point in $\mathbb{R}$. We call these lines the hyperbolic lines in H.

Lemma. 4.2
Through any two points $z_1, z_2 \in H$, there is a unique hyperbolic line l.

Proof. This is clear when $\Re z_1 = \Re z_2$. If not, then the perpendicular bisector of $z_1 z_2$ intersect $\mathbb{R}$ at one point, which is the centre of the semicircle.



□

Lemma. 4.3
$PSL(2,\mathbb{R})$ acts *transitively* on the set of hyperbolic lines.

Proof. It suffices to show that for all hyperbolic lines l, there exists $g \in PSL(2,\mathbb{R})$ s.t. $g(l) = L^+$. This is clear when l is a vertical half line. If l is a semicircle, endpoints $s < t \in \mathbb{R}$, then $g(z) = \frac{z-t}{z-s}$ which is valid as the determinant of the corresponding matrix is positive. Also, $g(t) = 0$, $g(s) = \infty$, and the only half line through them is L^+. □

Remark. Furthermore, we can achieve $g(s) = 0$, $g(t) = \infty$ by composing with $z \to -1/z$. Also we can map all given point $P \in l$ to $g(P) = i \in L^+$ (compose with $z \to az$, $a > 0$).

Definition. Given two points $z_1, z_2 \in H$, the *hyperbolic distance*, $\rho(z_1, z_2)$, is the length of segment $[z_1, z_2] \subset l$ of the unique hyperbolic line through z_1, z_2. Then $PSL(2,\mathbb{R})$ preserves ρ (by Lemma 4.2, Proposition 4.1 and some previous theory).

Proposition. 4.4
If $\gamma : [0,1] \to H$ is piece-wise C^1-norm with $\gamma(0) = z_1,$, $\gamma(1) = z_2$, then $length(\gamma) \geq \rho(z_1, z_2)$ with equality holds iff γ is the hyperbolic line through z_1 and z_2 parameterized monotonically (i.e. no going back).

Proof. We assume γ is C^1. $\exists\ g \in PSL(2,\mathbb{R})$ that takes $g(l) to L^+$ (which is an isometry). So WLOG let $z_1 = iu$, $z_2 = iv$, $u < v \in \mathbb{R}$. Then write

$\gamma(t) = x(t) + iy(t)$, we have

$$\begin{aligned} length(\gamma) &= \int_0^1 \frac{1}{y}\sqrt{\dot{x}^2+\dot{y}^2}dt \\ &\geq \int_0^1 \frac{|\dot{y}|}{y}dt \\ &\geq \left|\int_0^1 \frac{\dot{y}}{y}dt\right| \\ &\geq \log y(t)|_0^1 \end{aligned}$$

Thus

$$\rho(z_1, z_2) = \log\frac{v}{u}$$

Equality holds only if $\dot{x} \equiv 0$, $\dot{y} \geq 0$, i.e. monotonic. □

Remark. This proposition implies triangle inequality for $\rho(\cdot,\cdot)$: $length(\gamma) = \rho(z_1, z_2) \leq \rho(z_1, z_3) + \rho(z_3, z_2)$, with equality iff $z_3 \in \gamma$.

Thus (H, ρ) is a metric space.

Now consider the Geometry of the disc model.

Recall $\zeta \in D \to z = \frac{1+\zeta}{1-\zeta} \in H$.
$z \in H \to \zeta = \frac{z-i}{z+i} \in D$.

So (i) $PSL(2,\mathbb{R}) \cong$ the group of Möbius transformations sending $|\zeta| = 1$ to itself and $D \to D$. Call this group G.
(ii) Hyperbolic lines in D are segments of circles meeting $|\zeta| = 1$ orthogonally including diameters.
(iii) G acts transitively on hyperbolic lines in D.
(iv) The length minimizing curves are segments of hyperbolic lines parameterized monotonically.

Let ρ denote the hyperbolic distance.

Lemma. 4.5
(i) Rotations $z \to e^{i\theta}z$ ($\theta \in \mathbb{R}$) are in G;
(ii) if $a \in D$, then $g(z) = \frac{z-a}{1-\bar{a}z}$ is in G.

Proof. It's easy to see as these are linear maps, $|e^{i\theta}z| = |z|$, $d(e^{i\theta}z)| = dz$ (recall the metric $\frac{4|dz|^2}{(1-|z|^2)^2}$).
(iii) g sends the set $\{|z| = 1\}$ to itself: if $|z| = 1$, then

$$|1 - \bar{a}z| = |\bar{z}(1 - \bar{a}z)| = |\bar{z} - \bar{a}| = |z - a| \neq 0$$

So $\left|\frac{z-a}{1-\bar{a}z}\right| = 1$, and $|z| = 1 \implies |g(z)| = 1$. Also $g(a) = 0$. □

Exercise. (c.f. Q9 sheet 2, Complex Analysis sheet 1)
We can show conversely that every element G is of the form $g(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}$ for some real θ and $|a| < 1$.

Proposition. 4.6
If $0 \leq r < 1$, then

$$\rho(0, re^{i\theta}) = \rho(0, r) = 2\tanh^{-1} r \qquad (*)$$

In general, for $z_1, z_2 \in D$,

$$\rho(z_1, z_2) = 2\tanh^{-1}\left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right|$$

Proof. The first equality of (*) is clear from lemma 4.5(i). For the second one, use $\gamma(t) = t$, $0 \leq t \leq r$, then from definition of length we get

$$\rho(0, r) = \int_0^r \frac{2dt}{1 - t^2} = 2\tanh^{-1} r$$

which gives the first part.

For the general case, let l be the unique hyperbolic line through z_1, z_2. Apply the isometry $g(z) = \frac{z - z_1}{1 - \bar{z}_1 z}$ (by lemma 4.5(ii)), we get $g(z_1) = 0$, so $g(l)$ is a segment of a diameter. We may further rotate about 0, and get $g(z_2) = r \in \mathbb{R}_+$. Thus

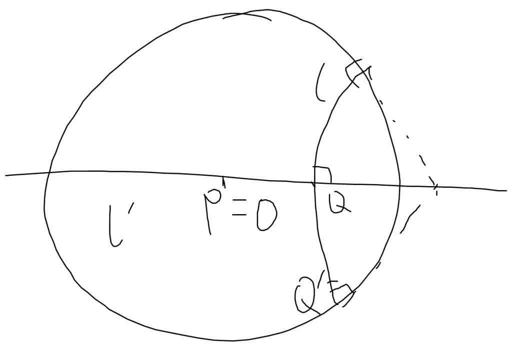
$$r = \left|\frac{z_2 - z_1}{1 - \bar{z}_1 z_2}\right|$$

and the proposition follows. □

Remark. When there is a 'distinguished' point, it's often convenient to map it to zero and use the Disc model.

Example. We show $\forall P$ and for all hyperbolic line l, $P \notin l$, there exists unique hyperbolic line l' s.t. l' meets l orthogonally, say $l \cap l' = Q$, and $\rho(P, Q) \leq \rho(P, Q')$ $\forall Q' \in l$.

WLOG let $P = 0 \in D$. Then just note the triangle inequality.

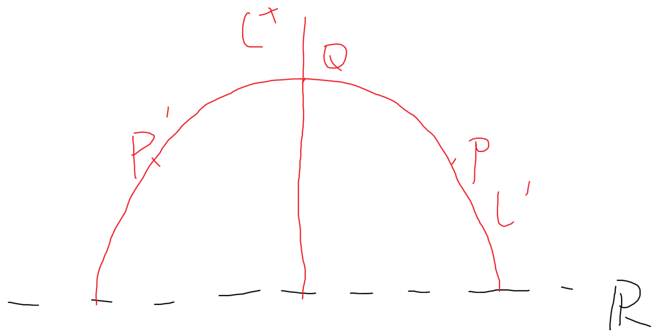


Lemma. 4.7
Suppose g is an isometry of H, and g fixes every point L^+. Then either $g = id_H$, or $g(z) - -\bar{z}$ $\forall z \in H$, i.e. a reflection in the $y-$axis.

Proof. Let $P \in H$, $P \notin L^+$. Then there is a unique line l' through P with $l' \perp L^+$, so l' is a semi-circle. Let $Q = l' \cap L^+$. Then

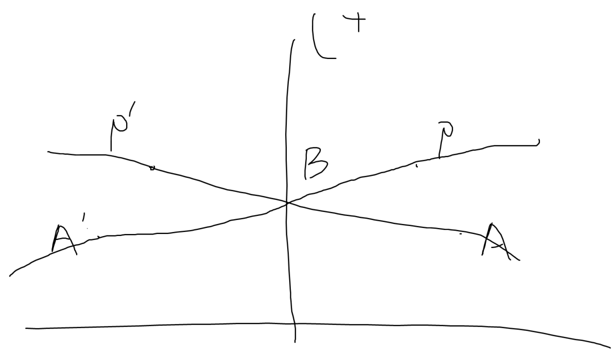
$$\rho(P, Q) = \rho(g(P), Q)$$

as $g(Q) = Q$.



Then $g(P) \in l'$ by the uniqueness of l', and either $g(P) = P$ or $g(P) = P'$, where P' is the image of P under the reflection $z \to -\bar{z}$. Now s.t.p. if $g(P) = P$, then $g = id_H$ (for if $g(P) = P'$ then compose g with $z \to -\bar{z}$ (an ieometry) to obtain g is $z \to -\bar{z}$).

Let $A \neq P$, $A \notin L^+$, $g(A) = A'$. WLOG let $P \in H^+ = \{z \in H | Re(z) > 0\}$. Let $A \in H^+$.



then $\rho(A', P) = \rho(A, P)$ (as g is isometry and $g(P) = P$). But $\rho(A', P) = \rho(A', B) + \rho(B, P) = \rho(A, B) + \rho(B, P)$, contradicts with triangle inequality $B \notin line(AP)$. Thus $g(A) = A$, i.e. g is identity. $\square$

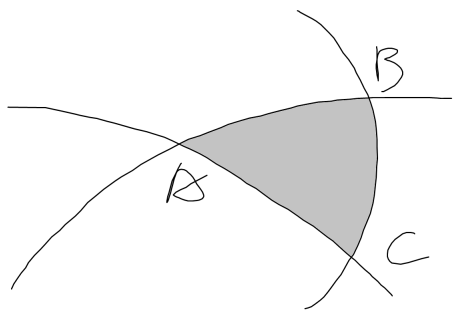
We call $R : z \in H \to -\bar{z} \in H$ the hyperbolic reflection in L^+, and for any hyperbolic line l in H with $T \in PSL(2, \mathbb{R}), T(l) = L^+$, call $R_l := T^{-1}RT$ the reflection (hyperbolic) in l.

By proposition 4.7, R_l is the unique isometry fixing points in l but is not the identity.

Exercise. Write out the reflections using the disc model.

4.3 Hyperbolic triangles

Definition. A hyperbolic triangle ΔABC is the region determined by 3 hyperbolic line segments.



Including cases when one vertex, say A, is at 'infinity', i.e. $A \in \mathbb{R} \cup \{\infty\}$ for H, $A \in \{|z| = 1\}$ for D, then $\alpha = 0$.

We shall prove that the area of $\Delta ABC = \pi - \alpha - \beta - \gamma$.

Theorem. 4.8 (Gauss-Bonnet for hyperbolic triangles)
For each hyperbolic triangle $T = \Delta ABC$ with angles $\alpha, \beta, \gamma \geq 0$,

$$area\ T = \pi - \alpha - \beta - \gamma.$$

Proof. First, do the case $\gamma = 0$, so C is at infinity. Use the H model, WLOG let $C = \infty$ (apply ($g \in PSL(2, \mathbb{R})$ if needed). Use $z \to z + a$, $a \in \mathbb{R}$, to centre the semicircle AB at 0 (noting AC, BC are in the vertical half-lines).

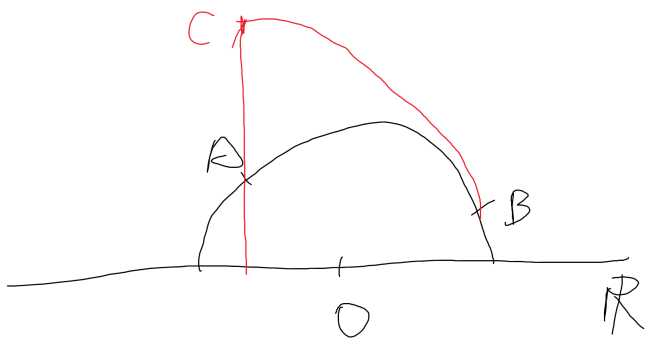
Use $z \to bz$ to arche the radius of semicircle of AB to be 1.

Thus WLOG $AB \subset \{x^2 + y^2 = 1, y > 0\}$ and then

$$\begin{aligned} \textit{area } T &= \int_{\cos(\pi-\alpha)}^{\cos\beta} \left(\int_{(1-x^2)^{1/2}}^{\infty} \frac{dy}{y^2} \right) dx \\ &= \int_{\cos(\pi-\alpha)}^{\cos\beta} \frac{dx}{(1-x^2)^{1/2}} \\ &= (-\arccos x)\big|_{\cos(\pi-\alpha)}^{\cos\beta} = (\pi-\alpha) - \beta \end{aligned}$$

noting $\arcsin x + \arccos x = \frac{\pi}{2}$, $\arccos : [-1, 1] \to [0, \pi]$, and as $\gamma = 0$.

In general, using the H model again, we can apply $g \in PSL(2, \mathbb{R})$ to move AC into a vertical line. Then as before move (with isometry) AB into a $\{x^2+y^2 = 1\}$ (AC will remain vertical).



Consider $\Delta_1 = AB\infty$, $\Delta_2 = BC\infty$. Then

$$\textit{area } \Delta_1 = \pi - \alpha - (\beta + \gamma), \textit{area } \Delta_2 = \pi - \delta - (\pi - \gamma)$$

So

$$\begin{aligned} area\ T &= area\Delta_1 - area\Delta_2 \\ &= \pi - \alpha - \beta - \delta - \pi + \delta + \pi - \gamma \\ &= \pi - \alpha - \beta - \gamma. \end{aligned}$$

□

There is hyperbolic version of sine and cosine rules (see Q16 sheet 2).

Every two lines on S^2 (i.e. great circles) meet, in 2 points; every two lines on $\mathbb{R}^2$ meet (in 1 point) if and only if they are not parallel.

Definition. Use the D model of hyperbolic plane, two hyperbolic lines l_1, l_2 are parallel iff they only meet at $\{|\zeta| = 1\}$, and are ultraparallel iff they do not meet anywhere in $\{\zeta| \leq 1\}$.

Euclid's parallel axiom (the 5th axiom) says that, given a line l and $P \notin l$, there exists unique line l' s.t. $P \in l'$ with $l \cap l' = \infty$. This fails both on S^2 and on the hyperbolic plane – but for a very different reason.

4.4 Thy hyperbolic model

Consider the *Lorenzian* inner product $\langle x, y \rangle$ on $\mathbb{R}^2$ with matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Set $q(\mathbf{x}) := \langle x, x \rangle = x^2 + y^2 - z^2$ for all $\mathbf{x} = (x, y, z)$. Let

$$S := \{\mathbf{x} \in \mathbb{R}^3 : q(\mathbf{x}) = -1\}$$

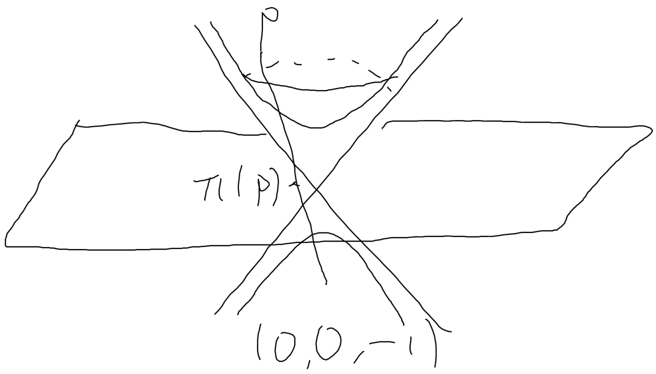
this is the 2-sheet hyperboloid, with

$$S^+ = S \cap \{z > 0\}$$

the upper sheet. Let $\pi : S^+ \to D \subset \mathbb{C}$ be

$$\pi(x, y, z) = \frac{x + iy}{1 + z} = u + iv$$

the stereographic projection from $(0, 0, -1)$.



Put $r^2 = u^2 + v^2$, and $\sigma = \pi^{-1} : D_{u,v} \to S^+ \subset \mathbb{R}^3$:

$$\sigma(u, v) = \frac{1}{1 - r^2}(2u, 2v, 1 + r^2)$$

Now check the inner product on the tangent plane to S^+ at $\sigma(u, v)$ spanned by $\sigma_n := \frac{\partial \sigma}{\partial u} = d\sigma(e_1)$, $\sigma_v = \frac{\partial \sigma}{\partial v} = d\sigma(e_2)$, e_1, e_2 are the standard basis of $\mathbb{R}^2$. Then

$$\sigma_u = \frac{2}{(1 - r^2)^2}(1 + u^2 - v^2, 2uv, 2u)$$
$$\sigma_v = \frac{2}{(1 - r^2)^2}(2uv, 1 + v^2 - u^2, 2v)$$

we restrict Lorenzian $\langle \cdot, \cdot \rangle$ to *span* $\langle \sigma_u, \sigma_v \rangle$ we get a symmetric bilinear form on $\mathbb{R}^2$ at each $(u, v) \in D$, $Edu^2 + 2Fdudv + Gdv^2$, with $E = \langle \sigma_u, \sigma_u \rangle = \frac{4}{(1-r^2)^2}$, $F = 0, G = E$,i.e.

5 Smooth embedded surfaces (in R^3)

Definition. Let $S \subset \mathbb{R}^3$. S is a *parameterised smooth embedded surface* if each $Q \in S$ has an open neighbourhood $Q \in U = W \cap S$ for W open in R^3 (subset topology) and a map $\sigma : V \to U$ from open $V \subset \mathbb{R}^2_{u,v}$ s.t.
- σ is a homomorphism of V onto U;
- $\sigma = \sigma(u, v)$ is C^∞ (all partial derivatives of all orders exist and are continuous);
- at each $Q = \sigma(P)$, the vectors $\frac{\partial \sigma}{\partial u}(P)$, $\frac{\partial \sigma}{\partial v}(P)$ are linearly independent.

Now $\sigma(u, v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$. Then

$$\sigma_u(P) = \frac{\partial \sigma}{\partial u}(P) = \begin{pmatrix} \partial x/\partial u \\ \partial y/\partial u \\ \partial z/\partial u \end{pmatrix} (P) = d\sigma_P(e_1), \sigma_v(P) = d\sigma_P(e_2)$$

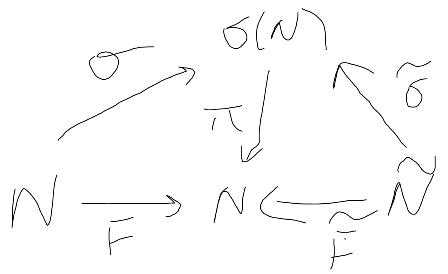
where e_1, e_2 are standard basis of $\mathbb{R}^2$. (u, v) are *smooth coordinates* on $U \subset S$.

The subspace *span*$_{\mathbb{R}}$ $\langle \sigma_u(P), \sigma_u(p) \rangle$ is the *tangent plane* $T_Q S$ to S at $Q = \sigma(P)$.

σ is a smooth (C^∞) parameterisation of $U \subset S$.

Proposition. 5.1
Suppose $\sigma : V \to U$, $\tilde{\sigma} : \tilde{V} \to U$ are two C^∞ parameterisations of U. Then the homomorphism $\varphi = \sigma^{-1} \circ \tilde{\sigma} : \tilde{V} \to V$ is a diffeomorphism.

Proof. It suffices to consider φ on a small neighbourhood of some $P = (u_0, v_0) \in \tilde{V}$. The Jacobi matrix of $\begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}$ has rank 2 for each $(u, v) \in V$ by the definition of σ. WLOG let (x_u, x_v) and (y_u, y_v) be linearly indepnedent at (u_0, v_0). Let $F(u, v) = \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix}$. Then by inverse function theorem (from Analysis II), F maps some open neighbourhood of $(u_0, v_0) \in N$ diffeomorphically onto the image (open) $N' \subset \mathbb{R}^2$. Now $\sigma(N)$ is open, $\tilde{N} \subset U = \tilde{\sigma}^{-1}(\sigma(N)) \subset \tilde{V}$ is open (by homomorphism property). $\sigma_1 F$ is bijective, so $\pi = F \circ \sigma^{-1}$ is also bijective. So $\tilde{F} = \pi \circ \tilde{\sigma}$.



Furthermore, $\pi(x,y,z) = (x,y)$ is certainly smooth since it's a linear map. Now $\varphi = \sigma^{-1} \circ \tilde{\sigma} = \sigma^{-1} \circ \pi^{-1} \circ \pi \circ \tilde{\sigma} = F^{-1} \circ \tilde{F}$ on $\tilde{N}$ a smooth map as F^{-1} and $\tilde{F}$ are so. By symmetry, φ^{-1} is also C^∞ on N. So done. □

Corollary. the tangent plane $T_Q S$ is independent of the choice of parameterisation σ.

Proof. Let $\tilde{\sigma}(\tilde{u},\tilde{v}) = \sigma(\varphi(\tilde{u},\tilde{v}), \varphi_2(\tilde{u},\tilde{v}))$, $\varphi = (\varphi_1, \varphi_2)$. By chain rule,

$$\tilde{\sigma}_{\tilde{u}} = (\varphi_1)_{\tilde{u}}\sigma_u + (\varphi_2)_{\tilde{u}}\sigma v, \tilde{\sigma}_{\tilde{v}} = (\varphi_1)_{\tilde{v}}\sigma_u + (\varphi_2)_{\tilde{v}}\sigma v$$

. Then the Jacobi matrix for φ is

$$J(\varphi) = \begin{pmatrix} \varphi_{1,\tilde{u}} & \varphi_{2,\tilde{u}} \\ \varphi_{1,\tilde{v}} & \varphi_{2,\tilde{v}} \end{pmatrix}$$

which is invertible as φ is a diffeomorphism. □

Remark. We can compute $\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}} = \det(J\varphi)\sigma_u \times \sigma_v$.

Definition. The *unit normal* to S at Q is

$$N = N_Q := \frac{\sigma_u \times \sigma_v}{||\sigma_u \times \sigma_v||}$$

Note that N is well-defined up to a sign.

$\theta := \sigma^{-1} : U \subset S \to V \subset \mathbb{R}^2$ is called a *chart.*

Example. Consider on S^2 the two stereographic projections from the North and South poles; they are both charts with domains covering S^2.

If $S \subset \mathbb{R}^3$ is an embedded surface, then each $T_Q S$ ($Q \in S$ inherits an inner product from $\mathbb{R}^3$ - i.e. we get a *family* of inner products depending on $Q \in S$. This family is the *first fundamental form of S*.

Given a parameterisation $\sigma : V \to U \subset S$ and $P \in V$, $a, b \in \mathbb{R}^2$, $\langle a, b\rangle_P := \langle d\sigma_P(a), d\sigma_P(b)\rangle_{\mathbb{R}^3}$ w.r.t. standard basis e_1, e_2 of $\mathbb{R}^2$, the RHS becomes $Edu^2 + 2Fdudv + Gdv^2$ with $E = \langle\sigma_u, \sigma_u\rangle_{\mathbb{R}^3}$, $F = \langle\sigma_u, \sigma_v\rangle_{\mathbb{R}^3}$, $G = \langle\sigma_v, \sigma_v\rangle_{\mathbb{R}^3}$. Here $\sigma_u = d\sigma(e_1)$, $\sigma_v = d\sigma(e_2)$.

This Riemannian metric of V is also called the first fundamental form w.r.t σ (especially in practical examples).

Fact: if $\tilde{\sigma} = \sigma \circ \varphi : \tilde{V} \to \tilde{U}$ as in proposition 5.1, then φ is an isometry of the respective Riemannian metric on V and $\tilde{V}$.

Definition. Given a smooth curve $\Gamma : [a,b] \to S \subset \mathbb{R}^3$,

$$length(\Gamma) := \int_a^b ||\Gamma'(t)||dt$$
$$energy(\Gamma) := \int_a^b ||\Gamma'(t)||^2 dt.$$

When $\Gamma([a,b]) \subset U = \sigma(V)$, then there exists unique $\gamma : [a,b] \to V$ open in $\mathbb{R}^2$ s.t. $\Gamma = \sigma \circ \gamma$ (we use these coordinates in $\mathbb{R}^2$ to express the curve in terms of u and v). So $\gamma = (\gamma_1, \gamma_2)$, $\Gamma'(t) = (d\sigma)_{\gamma(t)}(\dot{\gamma}_1(t)e_1 + \dot{\gamma}_2(t)e_2) = \dot{\gamma}_1\sigma_u + \dot{\gamma}_2\sigma v$. So

$$length(\Gamma) = \int_a^b \left(E\dot{\gamma}_1^2 + 2F\dot{\gamma}_1\dot{\gamma}_2 + G\dot{\gamma}_2^2\right)^{1/2} dt$$

Definition. Given a C^∞ parameterisation $\sigma : V \to U \subset S$ of surface S and a region $T \subset U$. Then

$$area(T) = \int_{\theta(T)} (EG - F^2)^{1/2} dudv$$

where $\theta(T) = \sigma^{-1}$ is the respective *chart*.

Proposition. 5.3
The area is well defined, i.e. $area(T)$ is independent of the partamerisation σ. Thus we may extend the definition of $area(T)$ to more general T which is not necessarily contained in one parameterized neighbourhood.

Remark. In practical examples, $\sigma(V) = U$ is often *dense* in S. Then it suffices to use just this U to compute $area(S)$.

Areas and lengths are invariant under isometries.

6 Geodesics

Let $V \subset \mathbb{R}^2_{u,v}$ open and we are given a Riemannian metric $Edu^2 + 2Fdudv + Gdv^2$. Suppose $\gamma = (\gamma_1, \gamma_2) : [a, b] \to V$ is a C^∞ curve.

Definition. γ is a *geodesic* if:
(1) $\frac{d}{dt}(E\dot{\gamma}_1 + F\dot{\gamma}_2) = \frac{1}{2}(E_u\dot{\gamma}_1^2 + 2F_u\dot{\gamma}_1\dot{\gamma}_2 + G_u\dot{\gamma}_2^2)$ and
(2) $\frac{d}{dt}(F\dot{\gamma}_1 + G\dot{\gamma}_2) = \frac{1}{2}(E_v\dot{\gamma}_1^2 + 2F_v\dot{\gamma}_1\dot{\gamma}_2 + G_v\dot{\gamma}_2^2)$
hold for all $t \in [a, b]$.

Let $\gamma(a) = p$, $\gamma(b) = q$. A *proper variation* of γ is a C^∞ map $h : [a, b] \times (-\varepsilon, \varepsilon) \to V \subset \mathbb{R}^2$ s.t. $h(t, 0) = \gamma(t)$, $t \in [a, b]$, $h(a, \tau) = p$, $h(b, \tau) = q$ for all $\tau \in (-\varepsilon, \varepsilon)$. So for all τ, $\gamma_\tau : [a, b] \to V$, $\gamma_\tau = h(t, \tau)$ is a C^∞ curve.

Proposition. 6.1
γ satisfies the geodesic ODEs iff γ is the stationary point of for the energy function for all proper variations, i.e. $\frac{d}{d\tau}|_{\tau=0}E(\gamma_\tau) = 0$.

Proof. We write $\gamma(t) = (i(t), v(t))$. Then

$$\begin{aligned} energy(\Gamma) &= \int_a^b \left(E(u,v)\dot{u}^2 + 2F(u,v)\dot{u}\dot{v} + G(u,v)\dot{v}^2\right) dt \\ &= \int_a^b I(u, v, \dot{u}, \dot{v}dt. \end{aligned}$$

Euler-Lagrange equations: a solution γ is stationary iff

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial I}{\partial \dot{u}}\right) &= \frac{\partial I}{\partial u}, \\ \frac{d}{dt}\left(\frac{\partial I}{\partial \dot{v}}\right) &= \frac{\partial U}{\partial v} \end{aligned}$$

But LHS of the first equation is just $2E\dot{u} + 2F\dot{v}$ and RHS is $e_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2$. So we get the first geodesic equation. The second is obtained similarly. □

Now let $S \subset \mathbb{R}^3$ be an embedded surface. $\sigma : V \to U \subset S$ a parameterisation, $\theta = \sigma^{-1} : U \to V$ the chart, and let $\Gamma : [a, b] \to S$ a smooth curve in S, $\gamma = \theta \circ \Gamma$ a smooth curve in V.

Define Γ to be a *geodesic* on S iff γ is a geodesic in V, i.e. iff Γ is a startionary point of $\int_a^b ||\Gamma'(t)||^2 dt$. This is independent of choice of σ.

Corollary. 6.2
If a curve Γ in S minimizes the energy among all the curves with the same end-points, then Γ is a geodesic.

Proof. Let $\Gamma : [a, b] \to S$. For all $a < a_1 < b_1 < b$, $\Gamma_1 = \Gamma|_{[a_1, b_1]}$ then minimizes the energy among all curves from $\Gamma(a_1)$ to $\Gamma(b_1)$.

If a_1, b_1 are such that $\Gamma[(a_1, b_1])\subset U$ for some parameterized neighbourhood, then Γ_1 must be a geodesic by proposition 6.1, Γ_1 is a geodesic. Now vary a_1, b_1 to get a cover of $[a, b]$. □

Lemma. 6.3
Let $V \subset \mathbb{R}^2$, $P, Q \in V$, V is endowed with a Riemannian metric. Consider C^∞ curve γ_0, $\gamma_0(0) = P$, $\gamma_0(1) = Q$. Then γ_0 minimizes the energy iff γ_0 minimizes the length and has constant speed $\dot{\gamma}_0$.

Proof. Cauchy-Schwartz for $f, g \in C[0, 1]$ says

$$\left(\int_0^1 fg\right)^2 \leq \int_0^1 f^2 \int_0^1 g^2$$

with equality attained iff $g = \lambda f$ for some $\lambda \in \mathbb{R}$, or alternatively $f = 0$.

Put $f \equiv 1$, $g = ||\dot{\gamma}||$. Then

$$(length(\gamma))^2 \leq energy(\gamma)$$

with equality attained only if $||\dot{\gamma}||$ is a constant.

If $length(\gamma) = l$, then the minimum of energy l^2 does occur exactly when $||\dot{\gamma}||$ is a constant. □

Remark. We can show that a curve γ is geodesic precisely if Γ locally minimizes energy, also iff γ locally minimizes length and has constant speed. By locally minimizing we mean that $\forall t_0$, $\exists \varepsilon > 0$ s.t. $\gamma|_{t_0-\varepsilon, t_0+\varepsilon]}$ minimizes length/energy.

Remark. Geodesic ODEs actually imply $||\Gamma'(t)||$ is a constant (see example sheet 3 Q7).

Further properties of the geodesics:

Recall that the defining ODEs are of the form

$$\frac{d}{dt}\left(\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}\right) = \text{ terms with derivative of lower order}$$

The matrix $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ is invertible, thus the ODE is of the form $(\ddot{u}, \ddot{v}) = \mathcal{F}(u, v, \dot{u}, \dot{v})$. Standard theory of ODEs (Analysis II, application of the contraction mappings) show that for all $P = (u_0, v_0) \in V \subset \mathbb{R}^2$, for all $\mathbf{a} = (p_0, q_0) \in \mathbb{R}^2$, there exists unique geodesic $\gamma(t) = (u(t), v(t))$, for $|t| < \varepsilon$, with $\gamma(0) = P$, $\dot{\gamma}(0) = \mathbf{a}$.

Example. Consider $S^2 \subset \mathbb{R}^3$, for all $P \in S^2$, all tangent direction (at P), there exists a unique great circle.

As arcs of great circles of length $< \pi$ are length minimizing, we find from Corollary 6.2 and Lemma 6.3, that the great circles are *all* the geodesics on S^2.

Similarly, on the hyperbolic plane, the hyperbolic lines are *all* the geodesics.

This can also be verified directly – see Q7 sheet 3.

We can use the geodesics on a surface $S \subset \mathbb{R}^3$ to construct around each point $P \in S$ the *geodesic polar coordinates* (a coordinate chart simplifying the coefficients of the first fundamental form(E, F, G)).

Sketch of proof:
Solutions of the geodesic ODEs depend on C^∞ on the initial conditions.
Let $\psi : U \to V \subset \mathbb{R}^2$ where V is open, and a coordinate chart $P \in U \subset S$ where U is open, and $\psi(P) = 0 \in V$.

For all value θ, there exists a unique geodesic $\gamma^\theta : (-\varepsilon, \varepsilon) \to V$ with $\gamma^\theta(0) = 0$, $\dot{\gamma}^\theta(0) =$the unit vector in the direction of $|tehta$.

Set $\sigma(r\theta) := \gamma^\theta(r)$. We can show:
1) σ is smooth in (r, θ);
2) For all θ_0, $\psi^{-1} \circ \sigma : \{(r, \theta) : 0 < r < \varepsilon, \theta_0 < \theta < \theta_0 + 2\pi\} := W \to S$, i.e. $\sigma : W \to V \setminus \{0\}$,
$psi^{-1} : V \setminus \{0\} \to U \setminus \{P\} \subset S$.

$\psi^{-1} \circ \sigma$ is a valid parameterisation, so $\sigma^{-1} \circ \psi$ is a valid *chart*.

The values (r, θ) of this chart are the *geodesic polar coordinates* at P.

Gauss lemma says the geodesic circles $\{r = r_0\} \subset W$ are perpendicular to their radii, i.e. to γ^θ, and the Riemmanian metric on W is

$$dr^2 + G(r, \theta) d\theta^2.$$

An *atlas* is a collection of charts (with domains) covering S. For example, geodesic polar coordinates define an atlas.

Other good atlases are given in sheet 3 (for $S = S^2$).

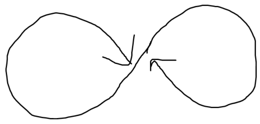
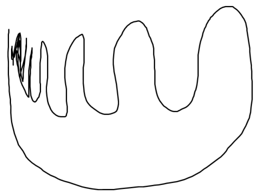
6.1 Surface of Revolution

We consider $S \subset \mathbb{R}^3$ that can be obtained by rotating a plane curve η around a straight line l.

WLOG let l be the $z-$axis and η in the $(x, z)-$plane, i.e.

$$\eta : (a, b) \subseteq \mathbb{R}, \eta(u) = (f(u), 0, g(u)).$$

We require:
(1) $||\eta'(u)|| = 1$ for all u. This basically requires the 'velocity' to be 1, and can be always obtained by parameterising using length;
(2) $f(u) > 0$;
(3) η is a homomorphism onto its image. This rules out some weird examples that we don't want, for example,

Define S as the image of $\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$, $a < u < b$, $0 \le v \le 2\pi$, and for all $\alpha \in \mathbb{R}$, $\sigma^\alpha : (a,b) \times (\alpha, \alpha + 2\pi)$ is a homomorphism onto its image (see Q1 sheet 3). Then

$$\sigma_u^\alpha = (f' \cos v, f' \sin v, g'),$$
$$\sigma_v^\alpha = (-f \sin v, f \cos v, 0)$$

so

$$\sigma_u \times \sigma_v = (-fg' \cos v, -fg' \sin v, ff'),$$
$$||\sigma_u^\alpha \times \sigma_v^\alpha||^2 = f^2(f'^2 + g'^2) = f^2 > 0 (\neq 0)$$

Thus σ^α is a valid parameterisation. so S is a valid embedded surface. The first fundamental form w.r.t. σ^α is

$$E = ||\sigma_u||^2 = f'^2 + g'^2 = 1,$$
$$F = \sigma_u \cdot \sigma_v = 0,$$
$$G = ||\sigma_v||^2 = f^2.$$

So the Riemannian metric is $du^2 + f^2 dv^2$.

Definition. Curves on S of the form $\gamma(t) = \sigma(t, v_0)$ are called *meridians*, $\gamma(t) = \sigma(u_0, t)$ are called *parallels*.

Then the geodesic ODEs for $\gamma = (u, v)$ in $V \subset \mathbb{R}^2$ are

$$\begin{cases} \ddot{u} & = f \cdot \frac{df}{du} \cdot \dot{v}^2 \\ \frac{d}{dt}(f^2 \dot{v}) & = 0 \end{cases}$$

Proposition. 6.4
Assume $||\dot{\gamma}|| = 1$, i.e. $\dot{u} + f^2(u)\dot{v}^2 = 1$. Then
(i) Every unit speed meridian $\gamma(t) = \sigma(t, v_0)$ is a geodesic;
(ii) A unit speed parallel $\gamma(t) = \sigma(u_0, t)$ is a geodesic precisely when $\frac{df}{du}(u_0) = 0$, i.e. u_0 is a stationary point.

Proof. (i) $v = v_0$ =constant. So the second equation holds. Also we have $\dot{u}$ is a constant since $\dot{v} = 0$. So the first equation holds as well.

(ii) $u = u_0$ =constant so $||\dot{\gamma}||^2 = f^2(u_0)\dot{v}^2 = 1$. So $\dot{v} = \pm\frac{1}{f(u_0)} \neq 0$ is a constant. Then the second equation holds. Now the first equation only holds if $\frac{df}{du}(u_0) = 0$ as $\ddot{u} = 0$. $\square$

7 Gaussian Curvature

Recall the curves $\eta : [0, l] \to \mathbb{R}^2$ a C^∞ curve with $||\eta'|| = 1$. Recall the curvature κ at $\eta(s)$ is determined by

$$\eta'' = \kappa \mathbf{n}$$

where $\mathbf{n}$ is a norm along η ($\mathbf{n} \cdot \eta' = 0$, $||$
$mathbfn = 1$, and $\kappa \geq 0$.

Let $f : [c, d] \to [0, l]$ be smooth, $f'(t) > 0$, so we may reparameterize $\gamma(t) = \eta(f(t))$. Then $\dot\gamma = \dot f \cdot \eta'(f(t))$, $||\dot\gamma||^2 = f^2$. Also $\eta''(f(t)) = \kappa\mathbf{n}$. $\kappa =$the curvature at $\gamma(t)$. By Taylor's theorem,

$$\gamma(t + \Delta t) - \gamma(t) = \dot f \cdot \eta'(f(t))\Delta t + \frac{1}{2}[\ddot f \cdot \eta'(f(t)) + \dot f^2 \cdot \eta''(f(t))](\Delta t)^2 + ...$$

So

$$\gamma(t + \Delta t) - \gamma(t)) \cdot \mathbf{n} = \frac{1}{2}||\dot\gamma||^2 \kappa (\Delta t)^2 + ...$$

$$\gamma(t + \Delta t) - \gamma(t)||^2 = ||\dot\gamma||^2 (\Delta t)^2 + ...$$

Thus $\frac{1}{2}\kappa =$ the ratio of the leading (quadratic) terms (above), and is independent of parameterisation.

Now let $\sigma : V \to U \subset S$ a parameterisation of surface $S \subset \mathbb{R}^3$. Apply Taylor's theorem,

$$\sigma(u+\Delta u, v+\Delta v) - \sigma(u, v) = \sigma_u \Delta u + \sigma_v \Delta v + \frac{1}{2}(\sigma_{uu}(\Delta u)^2 + 2\sigma_{uv}\Delta u \Delta v + \sigma_{vv}(\Delta v)^2) + ...$$

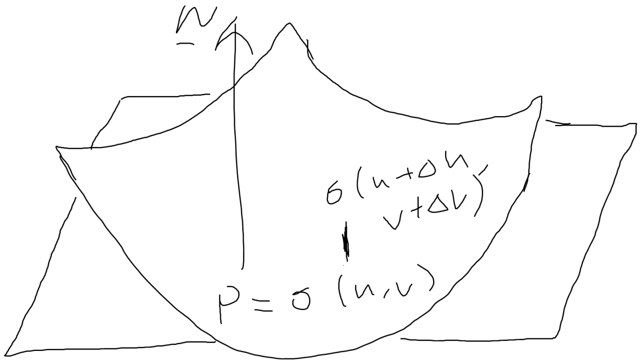
Recall

$$\mathbf{N} = \frac{\sigma_u \times \sigma_v}{||\sigma_u \times \sigma_v||}$$

Deviation from the tangent plane is

$$(\sigma(u + \Delta u, v + \Delta v) - \sigma(u, v)) \cdot \mathbf{N} = \frac{1}{2}(L(\Delta u)^2 + 2M\Delta u Deltav + N(\Delta v)^2) + ...$$

where $L = \sigma_{uu}\mathbf{N}$, $M = \sigma_{uv}\mathbf{N}$, $N = \sigma_{vv}\mathbf{N}$.



Recall

$$||\sigma(u+\Delta u, v+\Delta v) - \sigma(u,v)||^2 = E(\Delta u)^2 + 2F(\Delta u)(\Delta v) + G(\Delta v)^2 + ...$$

Definition. The second fundamental form on V (for S) is

$$Ldu^2 + 2Mdudv + Ndv^2$$

with $L, M, N \in C^\infty(N)$ as just defined.

Definition. The *Gaussian curvature* $\mathcal{K}$ of S at P is

$$\mathcal{K} = \frac{LN - M^2}{EG - F^2}$$

If $\mathcal{K} > 0$, the second fundamental form is either positive definite or negative definite.
On the other hand, if $\mathcal{K} < 0$, then the second fundamental form is indefinite.
If $\mathcal{K} = 0$, the second fundamental form is semi-definite.

Example. The unit sphere has $\mathcal{K} > 0$, the Pringle crisp has $\mathcal{K} < 0$.

Remark. It can be checked, similar to the curves story, that $\mathcal{K}$ does not depend on parameterisation.

Proposition. 7.1
Write $\mathbf{N}$ for the unit normal

$$\frac{\sigma_u \times \sigma_v}{||\sigma_u \times \sigma_v||}$$

Then at each point, $\mathbf{N}_u = a\sigma_u + b\sigma_v$, $\mathbf{N}_v = c\sigma_u + d\sigma_v (*)$, where

$$-\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix} \qquad (**)$$

in particular, $\mathcal{K} = ad - bc$.

Proof. $\mathbf{N} \cdot \mathbf{N} = 1$, so $\mathbf{N} \cdot \mathbf{N}_u = 0$ and $\mathbf{N} \cdot \mathbf{N}_v = 0$. So (*) holds for some a, b, c, d.

$$\begin{aligned} &\mathbf{N} \cdot \sigma_u = 0 \\ &\implies \mathbf{N}_u \cdot \sigma_u + \mathbf{N} \cdot \sigma_{uu} = 0 \\ &\implies \mathbf{N}_u \cdot \sigma_u = -L \end{aligned}$$

similarly, $\mathbf{N}_u \cdot \sigma_v = -M = \mathbf{N}_v \cdot \sigma_u$, $\mathbf{N}_v \cdot \sigma_v = -N$ dot (*) with σ_u and with σ_v, we get

$$\begin{aligned} -L &= aE + bF, \\ -M &= cE + dF \end{aligned}$$

$$\begin{aligned} -N &= aF + bG, \\ -N &= cF + dG \end{aligned}$$

which is (**). Take the determinants to obtain

$$\mathcal{K} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

□

Theorem. 7.2
Suppose for a $\sigma : V \to U \subset S \subset \mathbb{R}^3$. The first fundamental form $du^2 + G(u,v)dv^2$ $(G \in C^\infty(v))$. Then

$$\mathcal{K} = \frac{-(\sqrt{G})_{uu}}{\sqrt{G}}$$

Proof. To show $K = -\frac{(\sqrt{G}_{uu}}{\sqrt{G}}$ when the first fundamental form (Riemannian metric) is of the form $du^2 + G(u,v)dv^2$, set $e = \sigma_u$, $f = \frac{\sigma_v}{\sqrt{G}}$, $\mathbf{N} = e \times f$ an orthonormal basis of $\mathbb{R}^3$ depending on (u,v) ($\sigma(u,v)$ is a parameterisaion as before).

$e \cdot e = 1 \implies e \cdot e_u = 0 \implies e_u = \alpha f + \lambda_1 N$.

Similarly, $e_v = \beta f + \lambda_2 N$, $f_u = -\tilde{\alpha} e + \mu_1 \mathbb{N}$, $f_v = -\tilde{\beta} e + \mu_2 N (+)$. Then $e \cdot f = 0 \implies e_u \cdot f + e \cdot f_u = 0 \implies \alpha = \tilde{\alpha}$. Similar calculation shows $\beta = \tilde{\beta}$. Now

$$\begin{aligned}
\alpha &= e_u \cdot f \\
&= \sigma_{ii} \cdot \frac{\sigma v}{\sqrt{G}} \\
&= \left[(\sigma_u \cdot \sigma_v)_u - \frac{1}{2}(\sigma_u \cdot \sigma_u)_u \right] \frac{1}{\sqrt{G}} \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\beta &= e_v \cdot f \\
&= \sigma_{uv} \cdot \frac{\sigma v}{\sqrt{G}} \\
&= \frac{1}{2} G_u / \sqrt{G} \\
&= (\sqrt{G})_u
\end{aligned}$$

Also from (+),

$$\begin{aligned}
\lambda_1 u_2 - \lambda_2 u_1 & \\
&= e_u \cdot f_v - e_v \cdot f_u \\
&= (e \cdot f_v)_u - (e \cdot f_u)_v \\
&= -\beta_u \\
&= -(\sqrt{G})_{uu}.
\end{aligned}$$

From Proposition 7.1,

$$\begin{aligned}
\mathbf{N}_u \times \mathbf{N}_v &= (ad - bc)\sigma_u \times \sigma_v \\
&= \mathcal{K} \sigma_u \times \sigma_v \\
&= \mathcal{K} \sqrt{G} (e \times f)
\end{aligned}$$

So by VC identities

$$\begin{aligned}
K\sqrt{G} &= (\mathbf{N}_u \times \mathbf{N}_v) \cdot (e \times f) \\
&= (\mathbf{N}_u \cdot e)(\mathbf{N} \cdot f) - (\mathbf{N}_u \cdot f)(\mathbf{N}_v \cdot e)
\end{aligned}$$

But

$$(N \cdot e)_u = 0 = N_u \cdot e + N \cdot e_u.$$

So the above equals

$$(N \cdot e_u)(N \cdot f_u) - (N \cdot f_u)(N \cdot e_v) = \lambda_1\mu_2 - \lambda_2\mu_1 \quad - (\sqrt{G})_{uu}$$

So done. □

Definition. An *Abstract smooth surface* S is a metric space (or Hausdorf topological space) with coflection of homeomorphism called *charts* $\theta_i : U_i \to V_i$ on open $V_i \subset \mathbb{R}^2$, s.t.
(i) $S \cup_i U_i$;
(ii) $\forall i, j$, $\varphi_{ij} = \theta_i \circ \theta_j^{-1} : \theta_j(U_i \cap U_j) \to \theta_i(U_i \cap U_j)$ is a diffeomorphism.

A *Riemmanian metric* on S is given by a Riemmanian metric on each $V_i = \theta_i(U_i)$ subject to compatibility condition

$$\langle d\varphi_P(\mathbf{a}), d\varphi_P(\mathbf{b})\rangle_{\varphi(P)} = \langle \mathbf{a}, \mathbf{b}\rangle_P$$

where $\varphi = \varphi_{ij}$, $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$.

Then length, areas, energy, geodesics, etc are all well-defined on S via charts and first fundamental form E, F, G using formulae as before.

It can be shown that for all $P \in S$, we can construct the geodesic polar coordinates $(\rho, \theta) = (u, v)$ around P s.t. metric is $du^2 + G(u, v)dv^2$.

Now we *define* the *curvature* at P to be

$$\mathcal{K} = -\frac{(\sqrt{G}_{uu}}{\sqrt{G}}.$$

Example. (i) $\mathbb{R}^2$ with $du^2 + dv^2$.
(ii) $S^2 \subset \mathbb{R}^3$ embedded surface – Q3 sheet 3.
(iii) D unit in $\mathbb{R}^2$ with $\frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}$ isometric to H with $\frac{dx^2+dy^2}{y^2}$.

N.B.
• just one char suffices for (i) and (ii);
• hyperbolic plane *cannot* be realized as embedded surface in $\mathbb{R}^3$ (theorem of Hilbert).

(i) $dx^2 + dy^2$, $G = 1$ shows that $\mathcal{K} = 0$.
(ii) $S^2 \subset \mathbb{R}^3$ – exercise Q1 Sheet 3. Use spherical polars (fix radius = 1), get

$$\sigma(\rho, \theta) = (\sin\rho\cos\theta, \sin\rho\sin\theta, \cos\rho),$$
$$d\rho^2 + \sin^2\rho d\theta^2$$

(First fundamental form). $\sqrt{G} = \sin\rho$, $\mathcal{K} \equiv 1$.
(iii) Hyperbolic disc. Change x, y to Euclidean polars (r, θ). Then

$$\frac{4(dx^2 + dy^2)}{(1 - (x^2 + y^2))^2} = \frac{4(d\rho^2 + \rho^2 d\theta^2}{(1 - \rho^2)^2}$$

Let $\rho = 2\tanh^{-1} r$. Hyperbolic metric becomes

$$d\rho^2 + \sinh^2 \rho d\theta^2,$$
$$\sqrt{G} = \sinh \rho$$

So $\mathcal{K} \equiv -1$.

Triangulations make sense for abstract surfaces S too when S is compact.

Set $e(S) = F - E + V$ the Euler Number.

Theorem. (Gauss-Bonnet)
(1) If the sides of triangle $\Delta = ABC$ are geodesic segments, then

$$\int_\Delta K dA = (\alpha + \beta + \gamma) - \pi$$

where α, β, γ are angles, $dA = \sqrt{EG - F^2} dudv$ in each chart. So
(2) If S is compact, then

$$\int_S K dA = 2\pi \cdot e(S).$$

this is called the global Gauss-Bonnet.