Electromagnetism

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1 Introduction and Definitions

1.1 Electric charges and currents

The *charge* of a particle is an intrinsic property (like mass) determining the strength of the EM forces it experiences. Charge is *quantized* (discrete), always being a multiple $m \in \mathbb{Z}$ of the electron charge $q = -e$, where

$$e = 1.60217662(1) \times 10^{-19} C$$

where C is Coulomb (SI unit of charge).

Charge can be positive, negative or zero (natural). Examples: electrons ($q = -e$), positrons ($q = e$), proton ($q = e$), neutron ($q = 0$).

The *charge density* $\rho(x, t)$ describes charge per unit volume.

For a single particle q at positron x', we have

$$\rho(x, t) = q\delta(x - x') \tag{1.1}$$

While for N particles,

$$\rho(x, t) = \sum_{i=1}^{N} q_i \delta(x - xi) \tag{1.2}$$

where q_i, x_i are charge and position for the i^{th} particle, and δ satisfying

$$\delta(x - x') = 0 \forall x \neq x' \tag{1.3}$$

$$\int_V \delta(x - x') d^3x = \begin{cases} 1 & x' \in V \\ 0 & \text{else} \end{cases} \tag{1.4}$$

Also,

$$\int f(x)\delta(x - x') d^3x = f(x') \tag{1.5}$$

We will also consider *continuous* distributions $\rho(x, t)$ because N is large (and in QM particles are described by continuous wave functions $\psi(x, t)$.

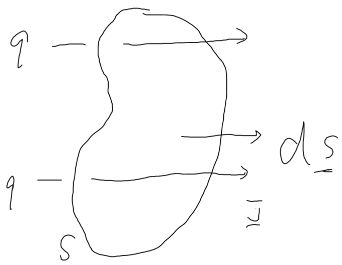
The *total charge* Q in a volume V is

$$Q = \int_V \rho(x, t) d^3x \tag{1.6}$$

Electric current, I describes the coherent motions of electric charge. We have

$$I = \frac{dQ}{dt} \tag{1.7}$$

across some surface S.



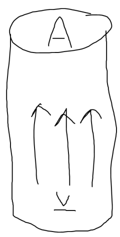
Like fluid flow, we define a *current density* $J(x,t)$, the rate charge passes across the surface element $d\mathbf{S} = \hat{\mathbf{m}} dS$ with normal $\hat{\mathbf{m}}$, that is, $dI = \mathbf{J} \cdot d\mathbf{S} = \mathbf{J} \cdot \hat{\mathbf{m}} dS$.

The total current across S is then

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} \tag{1.8}$$

Base SI unit of current is Ampere(A, $= Cs^{-1}$).

Example. Current in a wire.



Assume $1A$ current in 1mm diameter copper wire (lying in $z-$direction).
Uniform charge density: $\rho = ne$;
Electron density of Cu: $n = 8 \times 10^{28} m^{-3}$;
Current density $\mathbf{J} = \rho \mathbf{v} = -env\hat{\mathbf{z}}$.

So the total current density is $d\mathbf{S} = \hat{\mathbf{z}} dS$. So

$$\begin{aligned} I &= \int_A \mathbf{J} \cdot d\mathbf{S} \\ &= -\int env\hat{\mathbf{z}} \cdot \hat{\mathbf{z}} dA \\ &= -env\pi R^2 \\ &= -10^4 vCm^{-1} \end{aligned}$$

But $I = 1A$. So $v = 10^{-4} ms^{-1}$.

1.2 Forces and Fields

The Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1.9}$$

describes how a particle of charge **q** moves under the influence of an electric field $E(\mathbf{x}, t)$ and a magnetic field $\mathbf{B}(\mathbf{x}, t)$. SI unit for **E** are force per unit charge ($NC^{-1} = kgms^{-2}C^{-1}$).

The ratio $[E/B] = [V]$ means units for B, Tesla, are linked to particle motion (or currents in a wire) ($T = NC^{-1}m^{-1}s = N/(Am) = N/(Cms^{-1}) = 10^4 Gauss$).

Conversely, particles create EM fields, e.g. a static charge Q at $\mathbf{r} = 0$ has

$$E(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} \tag{1.10}$$

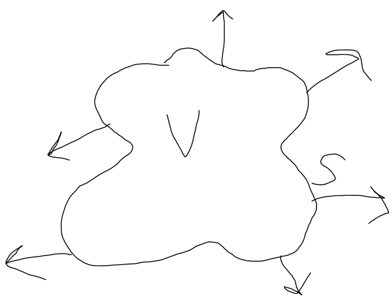
where the electric constant $\varepsilon_0 = 8.83 \times 10^{-11} C^2 kg^{-1} m^{-3} s^{-2}$ ($C^2 N^{-1} m^{-2}$). Also note $\varepsilon_0 = \frac{1}{\mu_0 c^2}$ (c is the speed of light) derives from the magnetic constant, where

$$\begin{aligned}\mu_0 &= 4\pi \times 10^{-7} C^{-2} kgm \\ &\approx 125 \times 10^{-6} C^{-2} kgm.\end{aligned}$$

Charge conservation is observed in every physical process. Charge Q in a volume V can only change by moving across a *closed surface* S, i.e.

$$\begin{aligned}\int_S \mathbf{J} \cdot d\mathbf{S} &= \int_V \nabla \cdot \mathbf{J} dx^3 \\ &= -\frac{dQ}{dt}\end{aligned} \tag{*}$$

The negative sign is because outward normals imply current flows *out* of V.



But from definition 1.6,

$$\frac{dQ}{dt} = \frac{d}{dt}\int_V \rho d^3x = \int_v \frac{\partial \rho}{\partial t} d^3x \tag{†}$$

So for arbitrary V, equation (*) and (†) must imply a local conservation law

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{1.11}$$

1.3 Maxwell's equations

All knowledge about the interplay between EM fields and particles is encoded in Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \text{(Gauss' Law, 1.12)}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(Gauss' Law for magnetism, 1.13)}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's law of induction, 1.14)}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \qquad \text{(Ampere-Maxwell Law, 1.15)}$$

2 Electrostatics

Consider time-independent charge distribution $\rho(\mathbf{x}, t)$ with $\mathbf{J} = 0$ (allowing us to set $\mathbf{B} = 0$ in equation (Gauss' Law for magnetism, 1.13) - (Ampere-Maxwell Law, 1.15)). Given ρ, we seek solutions of

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{2.1}$$

and

$$\nabla \times \mathbf{E} = 0. \tag{2.2}$$

2.1 Gauss' Laws

Integrate (2.1) over a volume V in $\mathbb{R}^3$ bounded by surface S:

$$\begin{aligned}\int_V \nabla \cdot \mathbf{E} d^3x &= \int_S \mathbf{E} \cdot d\mathbf{S} \\ &= \frac{1}{\varepsilon_0} \int_V \rho d^3x \\ &= \frac{Q}{\varepsilon_0}\end{aligned}$$

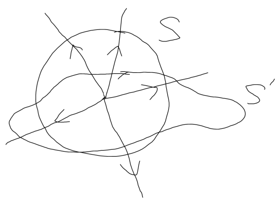
by divergence theorem, (2.1) and (1.6) respectively.

This implies Gauss' Laws

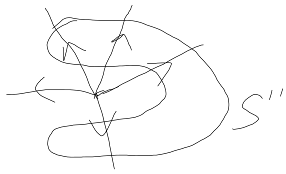
$$\Phi_{flux} = \int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\varepsilon_0} \tag{2.3}$$

where the middle is flux through S and the right is total charge in V.

For example, in the diagram below, we have $\Phi_S = \Phi_{S'}$ since they both enclose the charge.



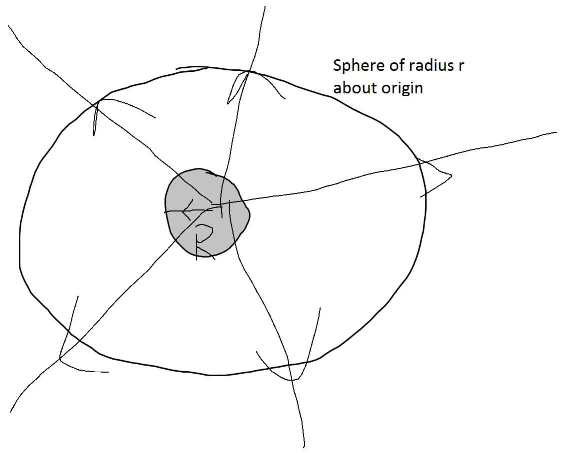
While in the diagram below $\Phi_{S''} = 0$ since the total charge in V is 0.



Gauss' Law can be used to find solutions for $\mathbf{E}$ in situations of spherical symmetry.

2.1.1 Application: Coulomb's Laws

Suppose we have spherically symmetric charge distribution $\rho(\mathbf{r}) = \rho(r)$ with $r = |\mathbf{r}|$, $\rho(r) = 0$ for $r > R$.



Here,

$$\int_0^R \rho 4\pi r^2 dr = Q$$

By symmetry, $\mathbf{E} = E(r)\hat{\mathbf{r}}$ (so $\nabla \times \mathbf{E} = 0$). Gauss Law yields

$$\begin{aligned}\int_S \mathbf{E} \cdot d\mathbf{S} &= \int_S E(r)\hat{\mathbf{r}} \cdot d\mathbf{S} \\ &= E(r)4\pi r^2 \\ &= \frac{Q}{\varepsilon_0}\end{aligned}$$

(Note $d\mathbf{S} = r^2 \sin\theta d\theta d\phi \hat{\mathbf{r}}$).

This implies Coulomb's Law (1.10),

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}, \ r > R \tag{2.4}$$

Interior solution($r < R$): Suppose uniform distribution in sphere, i.e.

$$\rho(r) = \begin{cases} \rho_0 & r \leq R \\ 0 & r > R \end{cases}$$

where ρ_0 is some constant. Then the total charge is

$$Q = \frac{4\pi}{3} R^3 \rho_0 \tag{*}$$

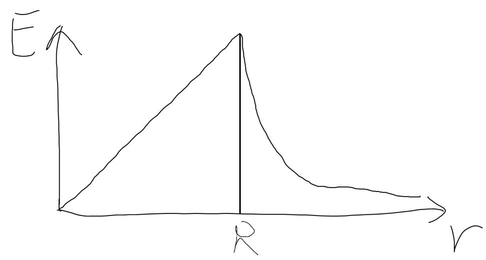
So

$$\begin{aligned}\int_S \mathbf{E} \cdot d\mathbf{S} &= E(r) 4\pi r^2 \\ &= \int_0^r \frac{\rho_0}{\varepsilon_0} 4\pi r^2 dr \\ &= \frac{1}{\varepsilon_0} \frac{4\pi}{3} r^3 \rho_0 \\ &= \frac{Q}{\varepsilon_0} (\frac{r^3}{R^3})\end{aligned}$$

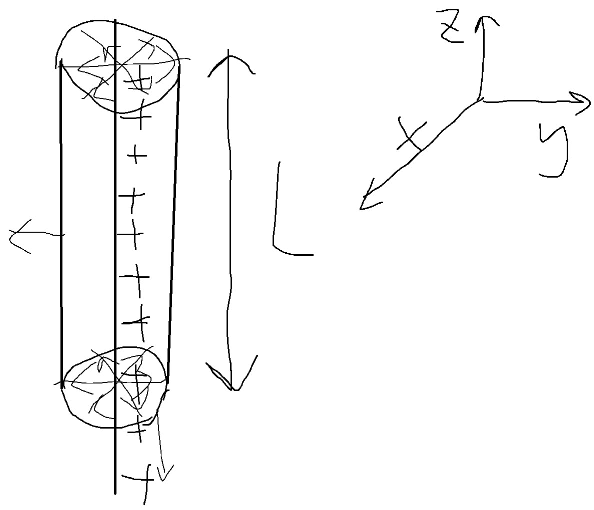
using (*). So

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{Qr}{R^3} \hat{\mathbf{r}} \ (r < R) \tag{2.5}$$

So the solution $\mathbf{E}$ looks like



Example. (Charged Line)
Consider a wire with constant charge η per unit length.



By symmetry in cylindrical polars,

$$\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}}$$

where $r = \sqrt{x^2 + y^2}$.

Take a cylinder of length L with $d\mathbf{S} = \hat{\mathbf{r}} r d\phi dz$ while upper and lower disks with $d\mathbf{S} = \pm\hat{\mathbf{z}} r dr d\phi$ – do not contribute as $\hat{\mathbf{z}} \cdot \hat{\mathbf{r}} = 0$. So

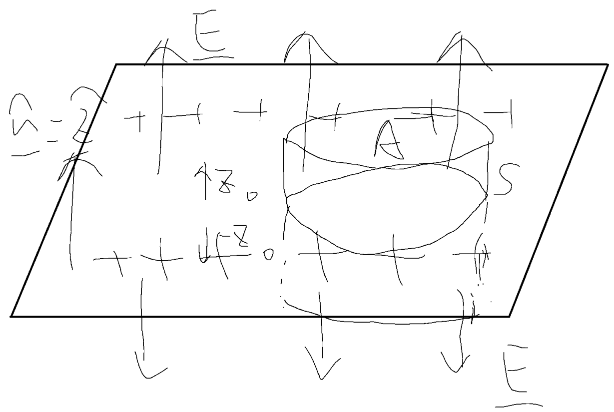
$$\begin{aligned}\int_S \mathbf{E} \cdot d\mathbf{S} &= E(r) 2\pi r L \\ &= \eta \frac{L}{\varepsilon_0}\end{aligned}$$

Hence we have our solution

$$E(r) = \frac{\eta}{2\pi\varepsilon_0} \frac{1}{r} \tag{2.5}$$

i.e. slower $\frac{1}{r}$ fall-off.

Example. (Surface charge and matching conditions)
Suppose we have a charged plane at $z = 0$. Charge density per unit area $\sigma(x, y)$.



By symmetry, we must have

$$E(\mathbf{r}) = E(z)\hat{\mathbf{z}}$$

and $E(z) = -E(-z)$.

Consider a small cylinder S enclosing an area A of the charged surface.

The surface integral becomes

$$\begin{aligned}\int \mathbf{E} \cdot d\mathbf{S} &= E(z_0)A \text{ (top disk)} - E(-z_0)A \text{ (bottom disk)} + \hat{\mathbf{z}} \cdot \hat{\mathbf{r}} \text{ (=0)} \\ &= 2E(z_0)A \\ &= \frac{\sigma A}{\varepsilon_0}\end{aligned}$$

by Gauss' Law ((2.3)) charge inside S.

Hence

$$E(z) = \frac{\sigma}{2\varepsilon_0} \ (z > 0) \tag{2.6}$$

which is independent of $z-$ perpendicular distance.

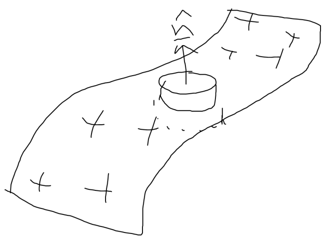
So as we approach the surface ($z \to 0$), there is a discontinuity

$$E(z \to 0^+) - E(z \to 0^-) = \frac{\sigma}{\varepsilon_0} \tag{2.7}$$

This result is easy to *generalize* to an arbitrary surface S (with normal $\hat{\mathbf{n}}$) with inhomogeneous σ:

$$\hat{\mathbf{n}} \cdot [\mathbf{E}^+ - \mathbf{E}^-] = \frac{\sigma}{\varepsilon_0} \tag{2.8}$$

This is the *matching condition* for the *normal component* of $\mathbf{E}$.

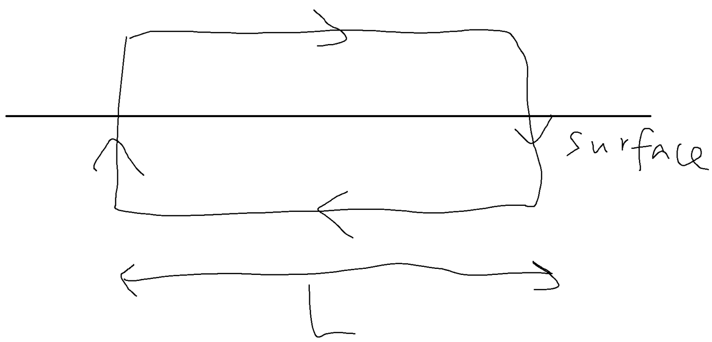


Aside: Any field $\mathbf{E}$ on S decomposes into normal $\mathbf{E}_\perp$ and tangential $\mathbf{E}_{//}$:

$$\mathbf{E} = \mathbf{E}_\perp + \mathbf{E}_{//} = (\mathbf{E} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + \mathbf{E} - (\mathbf{E} \cdot \mathbf{n})\hat{\mathbf{n}} = (\mathbf{E} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + \hat{\mathbf{n}} \times (\mathbf{E} \times \hat{\mathbf{n}})$$

We can show that the tangential component $\mathbf{E}_{//}$ on S has to be continuous:

$$\hat{\mathbf{n}} \times [\mathbf{E}^+ - \mathbf{E}^-] = 0 \tag{2.9}$$

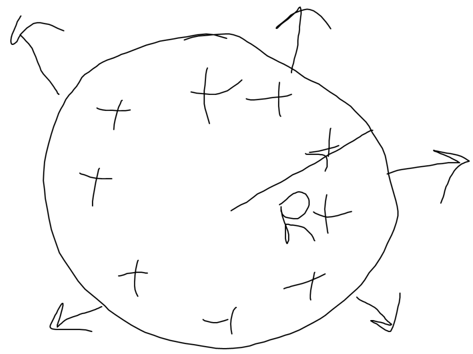


This can be shown using the line integral

$$\begin{aligned}\oint \mathbf{E} \cdot d\mathbf{l} &= \int_S \nabla \times \mathbf{E} \cdot d\mathbf{S} \\ &= 0\end{aligned} \tag{2.10}$$

By Stokes' Theorem and (2.2) respectively (see David Tong's lecture notes).

Example. Consider empty shell of radius R with surface charge σ with $Q = 4\pi R^2\sigma$.



This has the Coulomb solution

$$\mathbf{E} = \begin{cases} \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} = \frac{r}{\varepsilon_0}\left(\frac{R}{r}\right)^2 & r > R \\ 0 & r < R \end{cases}$$

So $\hat{\mathbf{r}} \cdot (\mathbf{E}^+ - \mathbf{E}^-) = \frac{\sigma}{\varepsilon_0}$ at $r = R$ on surface S satisfying (2.6) - (2.7).

2.2 Electrostatic potential

The scalar field is curl-free, i.e. $\nabla \times \mathbf{E} = 0$, so it can be expressed as the gradient of a scalar potential $\phi(\mathbf{x})$:

$$\mathbf{E} = -\nabla\phi \tag{2.11}$$

(Helmholtz decomposition: every differentiable vector field $\mathbf{F}$ can be decomposed into a curl-free and a divergence-free part, i.e. $\mathbf{F} = -\nabla\phi + \nabla \times \mathbf{A}$).

The electrostatic equation (2.1) $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$ becomes the *Poisson equation*

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0} \tag{2.12}$$

If $\rho = 0$, the homogeneous form is the *Laplace equation*

$$\nabla^2\phi = 0 \tag{2.13}$$

Both (2.12),(2.13) are linear, so we can use the *superposition* of solution (e.g. $\rho = \rho_1 + \rho_2$, $\phi = \phi_1 + \phi_2$ and $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$) notably by employing summation over Green's functions.

2.2.1 Point charge revisited

Consider (1.1) $\rho(\mathbf{r}) = q\delta(\mathbf{r})$, a point q at $\mathbf{r} = 0$.

$$\nabla^2\phi = -\frac{q}{\varepsilon_0}\delta(\mathbf{r}) \tag{2.14}$$

The symmetric solution must have

$$\phi(\mathbf{r}) = \phi(r)$$

so use Gauss' Law on the interior of a sphere S^2 centred at $\mathbf{r} = 0$. (normal $\hat{\mathbf{n}} = \hat{\mathbf{r}}$):

$$\begin{aligned}\int_V \nabla^2\phi d^3\mathbf{r} &= \int_{S^2} \nabla\phi\cdot\hat{\mathbf{n}}(=\frac{d\phi}{dr})dS \\ &= \frac{-q}{\varepsilon_0}\end{aligned}$$

from (2.14) with (1.4).
Hence, $4\pi r^2\frac{d\phi}{dr} = -\frac{q}{\varepsilon_0}$, $\frac{d\phi}{dr} = \frac{-q}{4\pi\varepsilon_0}\frac{1}{r^2}$ yielding

$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} + \text{ const} \tag{2.15}$$

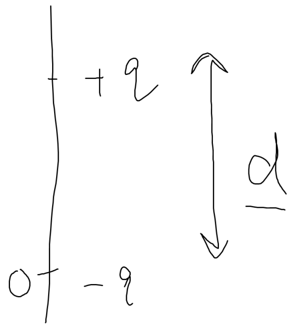
Assuming boundary conditions $\phi \to 0$ as $r \to \infty$, then this is free-space Green's function (check Methods).

As before (1.10), the electric field is

$$\mathbf{E} = -\nabla\phi = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}.$$

2.2.2 The Dipole

Consider two particles with opposite charge with $-q$ at $\mathbf{r} = 0$ and $+q$ at $\mathbf{r} = \mathbf{d}$.



By superposition of (2.15), the solution is simply

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{-q}{r} + \frac{q}{|\mathbf{r}-\mathbf{d}|}\right) \tag{2.16}$$

Expanding in Taylor series

$$f(\mathbf{r}-\mathbf{d}) = f(\mathbf{r}) - \mathbf{d}\cdot\nabla f(\mathbf{r}) + \frac{1}{2}(\mathbf{d}\cdot\nabla)^2 f(\mathbf{r}) + ...$$

So

$$\begin{aligned}\frac{1}{|\mathbf{r}-\mathbf{d}|} &= \frac{1}{r} - \mathbf{d}\cdot\nabla\left(\frac{1}{r}\right) + \frac{1}{2}(\mathbf{d}\cdot\nabla)^2\left(\frac{1}{r}\right) + ...\\ &= \frac{1}{r} + \frac{\mathbf{d}\cdot\mathbf{r}}{r^3} - \frac{1}{2}\left(\frac{|\mathbf{d}|^2}{r^3} - \frac{3(\mathbf{d}\cdot\mathbf{r})^2}{r^5}\right) + ...\end{aligned}$$

So the dipole solution as $|\mathbf{r}| \to \infty$

$$\begin{aligned}\phi &= \frac{q}{4\pi\varepsilon_0}\left(-\frac{1}{r} + \frac{1}{r} + \frac{\mathbf{d}\cdot\mathbf{r}}{r^3} + ...\right)\\ &\approx \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{d}\cdot\mathbf{r}}{r^3}.\end{aligned}$$

Defining the *dipole moment* $\mathbf{p} = q\mathbf{d}$ pointing from the negative to the positive charge, we have

$$\phi = \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2} \tag{2.17}$$

The dipole electric field is

$$\begin{aligned}\mathbf{E} = -\nabla\phi &= -\frac{1}{4\pi\varepsilon_0}\left((\mathbf{p}\cdot\mathbf{r})\nabla\left(\frac{1}{r^3}\right) + \frac{\nabla(\mathbf{p}\cdot\mathbf{r})}{r^3}\right)\\ &= \frac{1}{4\pi\varepsilon_0}\left(\frac{3(\mathbf{p}\cdot\mathbf{r})\hat{\mathbf{r}} - \mathbf{p}}{r^3}\right)\end{aligned} \tag{2.18}$$

2.2.3 General Green's function solution

Recall that the Laplacian $\nabla^2 G(\mathbf{r};\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$ has the free-space Green's function

$$G(\mathbf{r},\mathbf{r}') = \frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{2.19}$$

If we take charge distribution $\rho(\mathbf{r}) \neq 0$ only in a compact region $V \subset \mathbb{R}^3$.

Then the general solution for the Poisson equation (2.12) is

$$\begin{aligned}\phi(r) &= \frac{-1}{\varepsilon_0}\int G(\mathbf{r};\mathbf{r}')\rho(\mathbf{r}')d^3\mathbf{r}' \\ &= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'\end{aligned} \tag{2.20}$$

with electric field

$$\begin{aligned}\mathbf{E}(\mathbf{r}) = -\nabla\phi &= -\frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{r}')\nabla_{\mathbf{r}}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)d^3\mathbf{r}' \\ &= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}'\end{aligned} \tag{2.21}$$

Aside: If instead, Dirichlet boundary conditions imposed in the near(?) domain, then

$$G(\mathbf{r};\mathbf{r}') = -\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} + H(\mathbf{r},\mathbf{r}')$$

where H is harmonic and chosen so that $G = 0$ on the boundary of V (see Methods).

At far distances beyond V (i.e. $\mathbf{r} \gg \mathbf{r}', \forall \mathbf{r}'$), the solution (2.20) becomes

$$\begin{aligned}\phi(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0}\int d^3\mathbf{r}'\rho(\mathbf{r}')\left(\frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} + \ldots\right) \\ &= \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{R}\cdot\hat{\mathbf{r}}}{r^2} + \ldots\right)\end{aligned} \tag{2.22}$$

where total charge is (1.6)

$$Q = \int d^3\mathbf{r}'\rho(\mathbf{r}')$$

and the average dipole moment is

$$\mathbf{p} = \int d^3\mathbf{r}'\mathbf{r}'\rho(\mathbf{r}') \tag{2.23}$$

We can continue to expand to quadruple and higher moments.

2.2.4 Equipotentials and field lines

Move particle (charge q) along path $\mathbf{l}$ from $\mathbf{r}$ to $\mathbf{r}'$ in a potential $\phi(\mathbf{r})$. Potential energy U given by work done against force $\mathbf{F} = q\mathbf{E} = -q\nabla\phi$:

$$\begin{aligned} U &= -\int_{\mathbf{r}}^{\mathbf{r}'} \mathbf{F} \cdot d\mathbf{l} \\ &= -q\int_{\mathbf{r}}^{\mathbf{r}'} \mathbf{E} \cdot d\mathbf{l} \\ &= q\int_{\mathbf{r}}^{\mathbf{r}'} \nabla\phi \cdot d\mathbf{l} \\ &= q[\phi(\mathbf{r}') - \phi(\mathbf{r})] \end{aligned} \tag{2.24}$$

i.e. the potential difference between $\mathbf{r}$ and $\mathbf{r}'$.

Recall that any line integral $\int \mathbf{E} \cdot d\mathbf{l}$ along a closed curve is 0 since $\mathbf{E}$ is conservative.

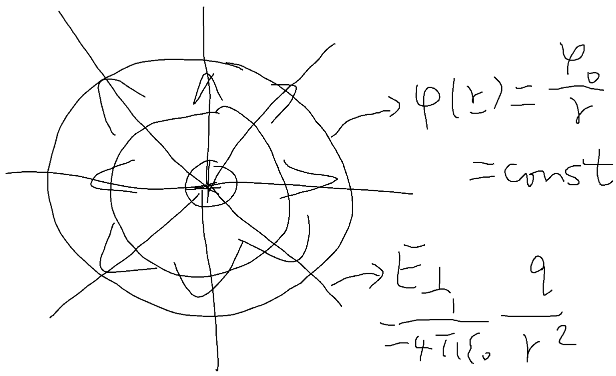
Suppose that $\phi(\mathbf{r}) = \phi(\mathbf{r}') = \phi_0$ along the path $\mathbf{l}$ from $\mathbf{r}$ to $\mathbf{r}'$, field satisfies

$$\phi(\mathbf{r}) = \phi_0 \tag{2.25}$$

which is a constant, thus defining a set of *equipotential surfaces*.

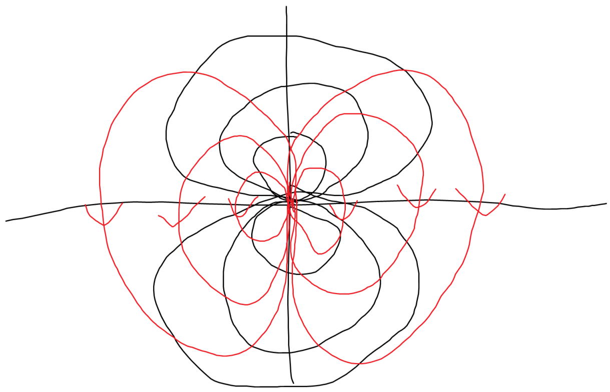
Since $\int \mathbf{E} \cdot d\mathbf{l} = 0$, this implies $\mathbf{E} = 0$ or $\mathbf{E}$ is normal to the surface. *Electric field lines* are continuous lines drawn tangent to $\mathbf{E}(\mathbf{r})$ with density proportional to $|\mathbf{E}|$.

Example. *Point charges* field lines begin at the positive charge and end at the negative charge.



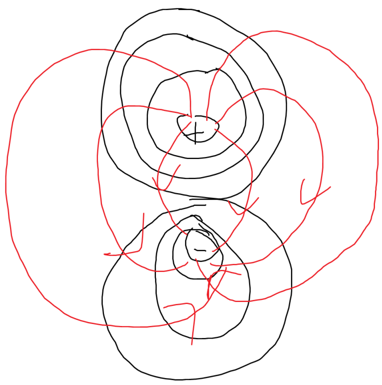
and similarly for the negative charge.

Example. Consider a pure dipole (2.17).



where $\phi = \frac{\rho\cos\theta}{4\pi\varepsilon_0 r^2}$, $r \sim (\cos\theta)^{1/2}$, $E = \frac{1}{4\pi\varepsilon_0 r^2}[2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}]$.

Example. The actual dipole is a combination of the above two.



2.3 Electrostatic energy

How much electrostatic energy do N charged particles have?

Place first particle (charge q_1, position $\mathbf{r}$) which creates potential

$$\phi_1(\mathbf{r}) = \frac{q_1}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}_1|}$$

but we do no work $W = 0$, i.e. we ignore the rest mass energy $E = mc^2$ of all particles (self energy).

Bring second particle from $r = \infty$ to $\mathbf{r} = \mathbf{r}_2$ yielding potential energy

$$W_2 = q[\phi_1(\mathbf{r}_2) - \phi_1(\infty)] = q_2\phi_1(\mathbf{r}_2) = \frac{q_1 q_2}{4\pi\varepsilon_0|\mathbf{r}_2-\mathbf{r}_1|}$$

Bring third particle from $r = \infty$:

$$\begin{aligned} W_3 &= q_3[\phi_2(\mathbf{r_2}) + \phi_1(\mathbf{r_1})] \\ &= q_3\left(\frac{q_2}{|\mathbf{r_3}-\mathbf{r_2}|} + \frac{q_1}{|\mathbf{r_3}-\mathbf{r_1}|}\right) \end{aligned}$$

etc.

Summing over all particles, the total PE is

$$\begin{aligned} U &= \sum_{i=1}^{N} W_i \\ &= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j>i}^{N}\frac{q_i q_j}{|\mathbf{r}_i-\mathbf{r}_j|} \\ &= \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{q_i q_j}{|\mathbf{r}_i-\mathbf{r}_j|} \\ &= \frac{1}{2}\sum_{i=1}^{N}\phi_i(\mathbf{r}_i) \end{aligned} \tag{2.26}$$

where we define $\phi_i(\mathbf{r}_i) = \frac{1}{4\pi\varepsilon_0}\sum_{j\neq i}\frac{q_j}{|\mathbf{r}_i-\mathbf{r}_j|}$.

Now consider the continuous limit of electrostatic energy (2.26):

$$U = \frac{1}{2}\int_V d^3\mathbf{r}\rho(\mathbf{r})\phi(\mathbf{r}) = \frac{\varepsilon_0}{2}\int_V d^3\mathbf{r}(\nabla\cdot\mathbf{E})\phi$$

by Maxwell's equation (2.1). This is then equal to

$$\frac{\varepsilon_0}{2}\int_V d^3\mathbf{r}[\nabla(\mathbf{E}\phi) - \mathbf{E}\cdot\nabla\phi] \tag{*}$$

using $\nabla\cdot(\mathbf{E}\phi) = \nabla\cdot\mathbf{E}\ \phi + \mathbf{E}\cdot\nabla\phi$.

But by divergence theorem

$$\int_V \nabla\cdot(\mathbf{E}\phi)d^3\mathbf{r} = \int_S \phi\mathbf{E}\cdot d\mathbf{S} \to 0$$

as $r\to\infty$, since on surface S, $\phi, E \to 0$ as $r\to\infty$ (for isolated charges $\phi\sim\frac{1}{r}, E\sim\frac{1}{r^2}, A\sim 4\pi r^2$, so $\int_S \to \frac{1}{r^3}r\pi r^2 \to 0$).

Using $\mathbf{E} = -\nabla\phi$, we find that (*) becomes

$$U = \frac{\varepsilon_0}{2}\int d^3\mathbf{r}\mathbf{E}\cdot\mathbf{E} \tag{2.27}$$

i.e. equivalent to the *energy of the electric field*.

2.4 Conductors

There are broadly 3 types of electrical materials: • *Insulators* have *bound electrons* with a large energy gap to the conduction band;
• *Semiconductors* have *limited* numbers of absent electrons ('holes') which can move;
• *Conductors* have *many free electrons* in a conduction band and current flows freely.

For *electrostatics* conductors have spherical properties:
• Any *interior electric field must vanish*: $\mathbf{E} = 0$, otherwise electrons would move.
• Since interior $\mathbf{E} = 0$, from $\mathbf{E} = -\nabla\phi$ we know ϕ =constant inside (equipotential).
• By $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$, there is no interior charge, i.e. $\rho = 0$ (despite there are many free electrons).
• Where have all the charges gone? They must all reside on the surface S (with normal $\hat{\mathbf{n}}$.
• describe with *surface charge density* σ.
• Any *electric field* E must be normal to conductor surface S (any tangential field $\mathbf{E}_{//}$ would move charges).

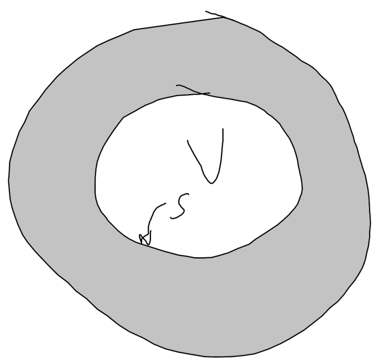
By matching conditions ((2.8)), the exterior field is

$$\mathbf{E} = \sigma/\varepsilon_0 \hat{\mathbf{n}} \tag{2.28}$$

i.e. Conductors define boundary conditions for the Poisson and Laplace's equations ((2.12),(2.13)).

2.4.1 Electrostatic shielding (Faraday cage)

The potential is constant inside the conductor ϕ_c =constant, so true also in the cavity (region V).



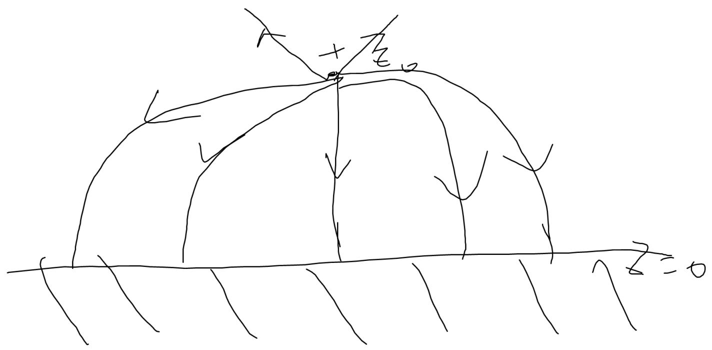
Since the potential satisfies Laplace's equation $\nabla^2\phi = 0$ we have $\mathbf{E} = 0$ inside cavity and no surface charge.

Exercise. Place charge Q inside and show that it will be shielded by equal opposite charge on S.

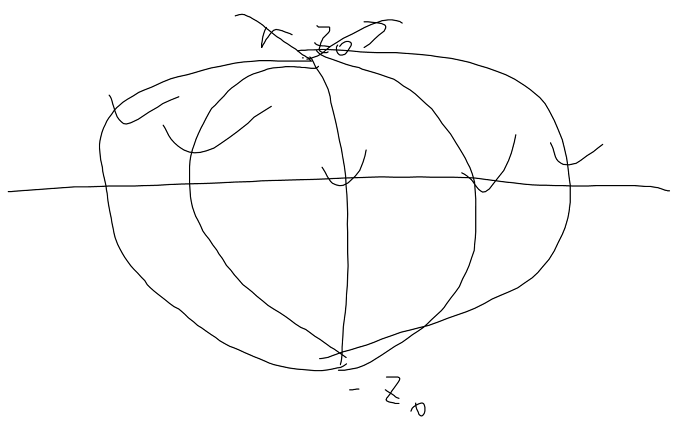
2.5 Method of images

Conductors provide equipotential boundary conditions which, if sufficiently symmetric, can be satisfied instead by adding additional image or mirror charges. The *uniqueness theorem* for solutions of Poisson's equation given ρ and boundary conditions on V means that any solution that satisfies these is the unique solution.

Example. Consider a point charge q, a distance z_0 from a conducting plane at $z = 0$ which is *grounded or earthed* (i.e. is held at $\phi = 0$).



Now instead of the conductor, place an image charge at $z = -z_0$.



The mirror solution is

$$\phi = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{x^2+y^2+(z-z_0)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+z_0)^2}}\right)$$

for which it is clear that $\phi = 0$ on the plane $z = 0$.

Thus we also have the unique solution for $z > 0$. Normal field E_z is given by

$$E_z = -\frac{\partial \phi}{\partial z} = \frac{q}{4\pi\varepsilon}\left(\frac{z - z_0}{(x^2 + y^2 + (z - z_0)^2)^{\frac{3}{2}}} - \frac{z + z_0}{(x^2 + y^2 + (z + z_0)^2)^{\frac{3}{2}}}\right) \quad (2.29)$$

This induces a surface charge at $z = 0$ from (2.28) $E = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$ given by

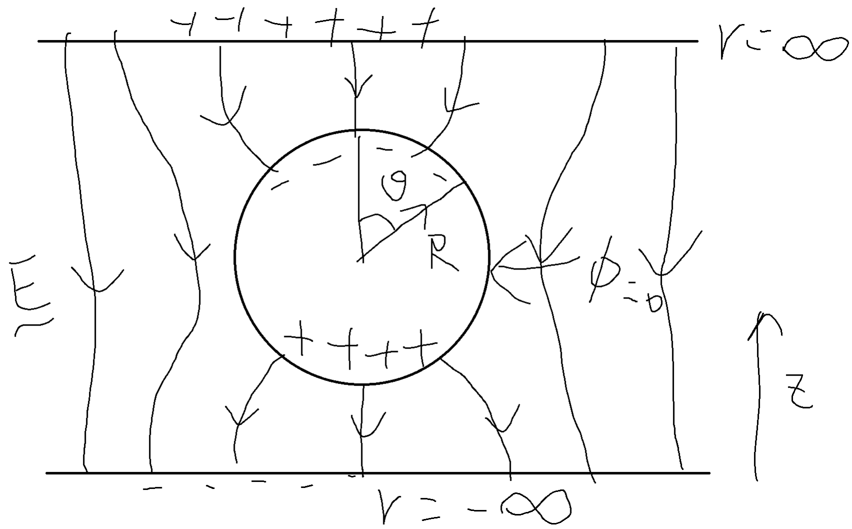
$$\sigma = \varepsilon_0 E_z|_{z=0} = \frac{q}{2\pi}\frac{z_0}{(x^2 + y^2 + z_0^2)^{\frac{3}{2}}} \quad (2.31)$$

Exercise. Show that the total induced surface charge

$$Q = \int dxdy\sigma = -q$$

i.e. the same as the image charge.

Example. (Conducting sphere in an electric field)



Initially uniform field $\mathbf{E} = -E_0\hat{\mathbf{z}}$ plus a conducting sphere (radius S, centre $\mathbf{r} = 0$, grounded $\phi = 0$).

Cylindrical symmetry $\mathbf{E}(\mathbf{r}) = \mathbf{E}(r, \theta)$ in 3D polar coordinates.

Instead of image charge, try adding image dipole field at $\mathbf{r} = 0$,

$$\phi(r, \theta) = -E_0\hat{\mathbf{z}} + \frac{\mathbf{p} \cdot \mathbf{r}}{4\pi\varepsilon r^2}$$

by (2.17). By syymetry, take $\mathbf{p} = p_0\hat{\mathbf{z}}$. So $\hat{\mathbf{r}} \cdot \hat{\mathbf{z}} = \cos\theta$.

We require $\phi = 0$ at $r = |\mathbf{r}| = R$, satisfied if

$$\phi(R, \theta) = -E_0 R\cos\theta + \frac{p\cos\theta}{4\pi\varepsilon_0 R^2} = 0$$

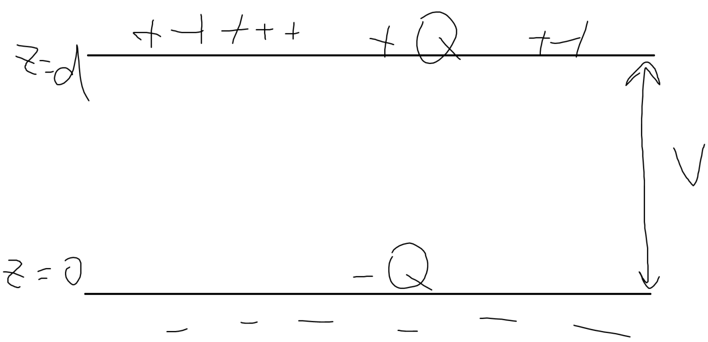
So
$$p = 4\pi\varepsilon_0 R^3 E_0$$
Solution for $r > R$ (by uniqueness) is
$$\phi(r,\theta) = -E_0 r\cos\theta + \frac{E_0 R^3\cos\theta}{r^2}$$
the first term is a uniform field, while the second term is a dipole field.

Exercise. Find $\mathbf{E} = -\nabla\phi$.

2.6 Capacitors

These are usually closely spaced conductors which store electrical energy. A potential difference V caused by opposite charges $\pm Q$ accumulating, with capacitance defined by
$$C = \frac{Q}{V} \tag{2.32}$$

Consider two parallel conductor plates with area A, charges $\pm Q$, positined at $z = 0, d$ (with $d \ll \sqrt{A}$).



$\mathbf{E} = -E_0\hat{\mathbf{z}} = -\frac{\sigma}{\varepsilon_0}\hat{\mathbf{z}}$ is a constant with $\sigma = Q/A$.

Since $\mathbf{E} = -\frac{d\phi}{dz}$, we must have $\phi(z) = E_0 z + c$, and potential difference $V = \phi(d) - \phi(0) = E_0 d = \frac{Qd}{A\varepsilon_0}$.

Hence, the capacitance is
$$C = \frac{A\varepsilon_0}{d} \tag{2.33}$$

The electrical energy (2.27) stored by a capacitor is
$$U = \frac{1}{2}\int d^3\mathbf{x}\mathbf{E}\cdot\mathbf{E} = \frac{\varepsilon}{2}Ad\left(\frac{Q}{A\varepsilon_0}\right)^2 = \frac{Q^2 d}{A\varepsilon} = Q^2 C \; (\frac{Q^2}{2C}??)$$

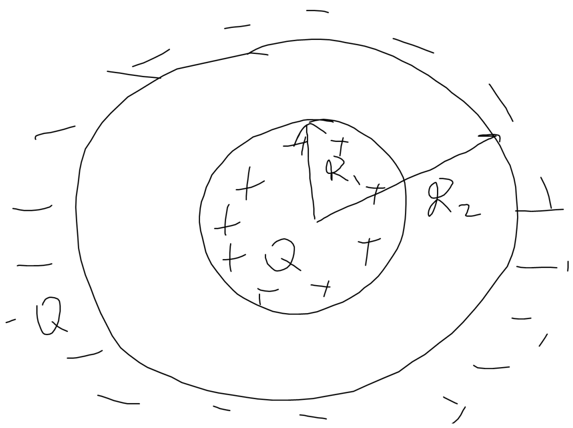
by (2.33).

Exercise. Spherical capacitor with $\pm Q$ at $R_1 < R_2$ with

$$\phi = \frac{Q}{4\pi\varepsilon_0 r}$$

for $R_1 < r < R_2$. Show that

$$C = \frac{4\pi\varepsilon_0 R_1 R_2}{R_2 - R_1}$$



3 Magnetostatics

We will now solve Maxwell's equations sourced by steady currents $J \neq 0$ which gives rise to magnetic fields $\mathbf{B}$. We will take $\rho = 0$, $\mathbf{E} = 0$, and $\frac{\partial \mathbf{J}}{\partial t} = 0$, so (Gauss' Law, 1.12) - (Ampere-Maxwell Law, 1.15) become

$$\nabla \times \mathbf{B} = \mu_0 J \tag{3.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.2}$$

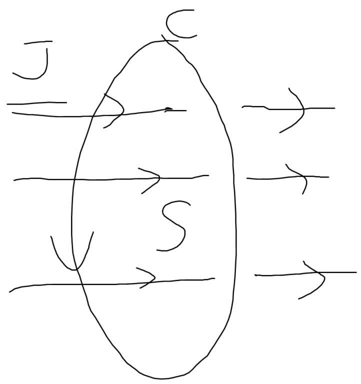
The continuity equation (1.11), $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$ implies that

$$\nabla \cdot \mathbf{J} = 0 \tag{3.3}$$

3.1 Ampere's Law

3.1.1 Straight wire with steady current

Suppose we have a steady current flowing through a surface S with boundary curve C, element $d\mathbf{l}$.



By Stokes' theorem,

$$\int_S \nabla \times \mathbf{B} \cdot dS = \oint_C \mathbf{B} \cdot d\mathbf{l}$$
$$= \mu_0 \int \mathbf{J} \cdot d\mathbf{S}$$

This is Ampere's Law

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \tag{3.4}$$

where I is the current through S.

Consider cylindrical coordinates (r, φ, z) with wire along $z-$axis and current I. By symmetry, we have $\mathbf{B}(\mathbf{r}) = B(r)\hat{\phi}$. This is the *right hand rule* - thumb points along current, fingers around B field lines.

Check (3.2):

$$\nabla \cdot \mathbf{B} = \frac{1}{r}\frac{\partial B(r)}{\partial \varphi} = 0$$

Around z =constant circle, we have

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int_0^{2\pi} B(r) r d\varphi = 2\pi r B(r) = \mu_0 I$$

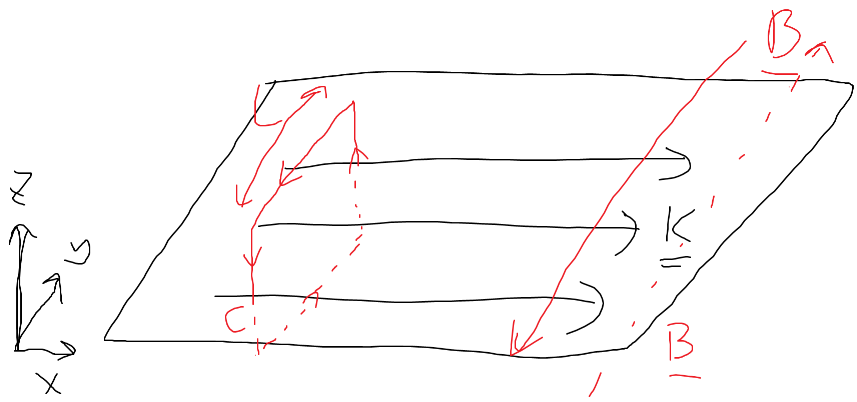
by Ampere's Law. So we have

$$B(r) = \frac{\mu_0 I}{2\pi r}\hat{\varphi} \tag{3.5}$$

Compare with line charge (2.6).

3.1.2 Surface currents and matching conditions

Suppose $z = 0$ plane has a steady current with current density $\mathbf{K} = k\hat{\mathbf{x}}$ (current per unit length).



By symmetry,

$$\mathbf{B} = \begin{cases} -B(z)\hat{\mathbf{y}} & z > 0 \\ B(-z)\hat{\mathbf{y}} & z < 0 \end{cases}$$

Now integrate about loop of length L in the x =constant plane, we have

$$\oint \mathbf{B}\cdot d\mathbf{l} = LB(z) - LB(-z)$$
$$= 2LB(z)$$
$$= \mu_0 kL$$

So we have

$$B(z) = \frac{mu_0 k}{2} \tag{3.6}$$

which is a constant field (compare with (2.7)).

Note the discontinuity across the surface $B(z \to 0^+) - B(z \to 0^-) = \mu_0 k$.

This can be generalized to the following matching conditions

$$\hat{\mathbf{n}} \times [\mathbf{B}^+ - \mathbf{B}^-] = \mu_0 k \tag{3.7}$$

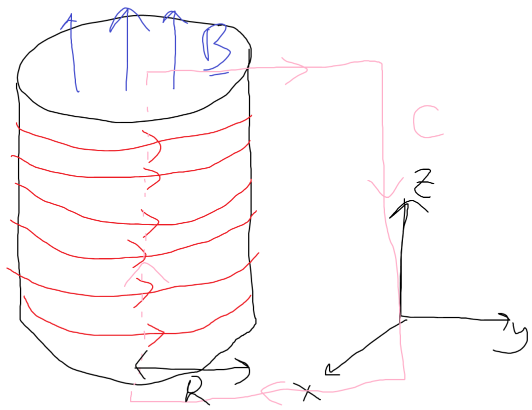
and

$$\hat{\mathbf{n}} \cdot [\mathbf{B}^+ - \mathbf{B}^-] = 0 \tag{3.8}$$

Note the duality with $\mathbf{E}$, see (2.8) -(2.9).

3.1.3 Solenoid

By wrapping wire continuously around a cylinder, we can create a circular surface current (infinite length).



By symmetry, $\mathbf{B} = b(r)\hat{\mathbf{z}}$ with $r = \sqrt{x^2+y^2}$.

Away from surface $\mathbf{J} = 0$, so by (3.1), $\nabla \times \mathbf{B} = 0$, so $\frac{dB}{dr} = 0 \implies B(r)$ =constant.

Consider the curve C: Outside $r > R$ we must have $B \equiv 0$, since physically $B \to 0$ as $r \to \infty$. Apply Ampere's law,

$$\int \mathbf{B}\cdot d\mathbf{l} = BL + 0 + 0 + 0 = BL$$
$$= \mu_0 INL$$

where I is the current in each wire the N is the number of winding per unit length. So we have

$$B = \mu_0 I N. \tag{3.9}$$

Check (3.7):

$$B = \begin{cases} \mu_0 I N \hat{\mathbf{z}} & r < R \\ 0 & r > R \end{cases}$$

So $\hat{\mathbf{n}} \times \Delta \mathbf{B} = \mu_0 \mathbf{K}$, where $\mathbf{K} = I N \hat{\mathbf{z}}$ which is consistent.

3.2 Vector potential

Recall from Methods the *Helmholtz theorem*, that any vector field $\mathbf{F}$ can be decomposed as

$$\mathbf{F} = \nabla \phi + \nabla \times \mathbf{A} \tag{3.10}$$

i.e. a curl-free (irrotational) part and a divergence-free (solenoidal) part, where $\mathbf{F} \to 0$ as $r \to \infty$,

$$\phi(\mathbf{r}) = \frac{1}{4\pi} \int \frac{\nabla_{\mathbf{r}} \cdot \mathbf{F}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

and

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi} \int \frac{\nabla_{\mathbf{r}'} \times \mathbf{F}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'.$$

For magnetostatics $\nabla \times \mathbf{B} = 0$, we can describe it with a vector potential $\mathbf{A}$,

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{3.11}$$

Now applies (3.1),

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$$

so

$$-\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = \mu_0 \mathbf{J}. \tag{3.12}$$

3.2.1 Gauge transformations

Note that $\mathbf{B}$ is unique, but $\mathbf{A}$ is not. Consider

$$\mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \nabla \chi(\mathbf{r}) \tag{3.13}$$

for some arbitrary smooth function χ.

Clearly $\nabla \times \mathbf{A}' = \nabla \times \mathbf{A}$.

3.2.2 Coulomb gauge

It is *often* convenient to choose χ s.t. $\nabla \cdot \mathbf{A}' = 0$.e In other words, we fix to the Coulomb gauge. Can we always do this?

Consider gauge transformation $\mathbf{A}' = \mathbf{A} + \nabla x$ yielding identical $\mathbf{B} = \nabla \times \mathbf{A}$. Suppose $\nabla \cdot \mathbf{A} = \psi(\mathbf{r}) \neq 0$, then

$$\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} + \nabla^2 \xi = \psi(\mathbf{r}) + \nabla^2 \xi = 0$$

if ξ satisfies Poisson's equation $\nabla^2 \psi = \psi(x)$ for which there is always a unique solution.

Exercise For the straight wire (3.5), verify that

$$\mathbf{A}(\mathbf{r}) = \frac{-\mu_0 I}{2\pi} \ln r \hat{\mathbf{z}}$$

and reproduces the correct magnetic field

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi r} \hat{\varphi} = \frac{\mu_0 I}{2\pi r} \left(-\frac{y}{r} \hat{\mathbf{x}} + \frac{x}{y} \hat{\mathbf{y}} \right)$$

(and is in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$).

3.3 Biot-Savart Law

Consider (3.12) in Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, so Maxwell equation (3.1) becomes

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = \mu_0 \mathbf{J}$$

So

$$\nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \tag{3.16}$$

or in components, for $i = 1, 2, 3$

$$\nabla^2 A_i = -\mu_0 J_i$$

which are 3 copies of Poisson equations. We've solve this already with Green's functions (2.20), implying

$$A_i(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V d^3\mathbf{x}' \frac{J_i(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

or

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V d^3\mathbf{x}' \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tag{3.17}$$

The magnetic field is then (using $\nabla \times (\psi \mathbf{D}) = \psi \nabla \times \mathbf{D} + \nabla \psi \times \mathbf{D}$),

$$\begin{aligned} \mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi} \int_V d^3\mathbf{x}' \nabla_{\mathbf{x}} \times \left(\frac{\mathbf{J}(\mathbf{x})}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &= \frac{\mu_0}{4\pi} \int d^3\mathbf{x}' \left[\frac{\nabla_{\mathbf{x}} \times \mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \nabla_{\mathbf{x}} \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \times \mathbf{J}(\mathbf{x}') \right] \end{aligned}$$

The first term is 0 as there is no $\mathbf{x}-$dependence, and the second term is equal to $\frac{-(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$. Hence, we have *Biot-Savart Law*,

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int d^3\mathbf{x}' \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \tag{3.18}$$

For localized current along a curve C (by straight wire with $\mathbf{J}(\mathbf{x}) = I\delta(x)\delta(y)\hat{\mathbf{z}}$), then (3.18) becomes

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\mathbf{x}' \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \tag{3.19}$$

Aside: verify $\mathbf{A}(\mathbf{x})$ in (3.17) is in Coulomb gauge $\nabla \cdot \mathbf{A} = 0$:

$$\begin{aligned}\nabla_{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi} \int_V d^3\mathbf{x}' \mathbf{J}(\mathbf{x}') \cdot \nabla_{\mathbf{x}} \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} \right) \\ &= \frac{-\mu_0}{4\pi} \int d^3\mathbf{x}' \mathbf{J}(\mathbf{x}') \cdot \nabla_{\mathbf{x}'} \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} \right)\end{aligned}$$

since if we interchange $\mathbf{x} \leftrightarrow \mathbf{x}'$,

$$\nabla_{\mathbf{x}} \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} \right) = -\nabla_{\mathbf{x}'} \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} \right)$$

and the above then equals

$$\frac{-\mu_0}{4\pi} \int_V d^3\mathbf{x}' \left[\nabla_{\mathbf{x}'} \left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \right) - \frac{\nabla_{\mathbf{x}'} \cdot \mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \right] = 0$$

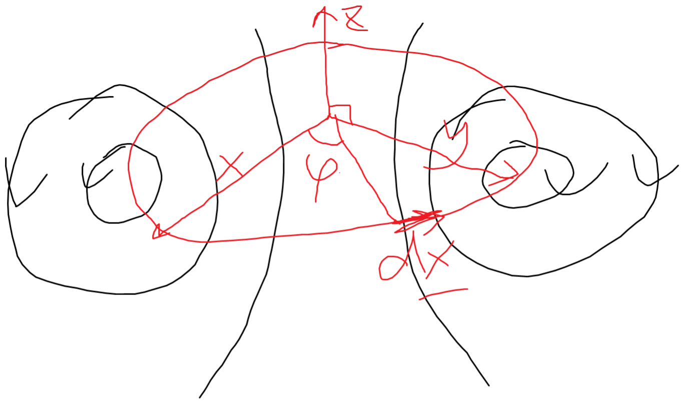
by (3.3) the continuity equation.

3.4 Current loop and magnetic dipole

Consider a circular loop, current I, radius R, lying in $z = 0$ plane. We could solve (3.17) directly with

$$\mathbf{J} = I \sin\theta \delta(\cos\theta) \frac{\delta(r-R)}{R} \times (-\sin\varphi \hat{\mathbf{i}} + \cos\varphi \hat{\mathbf{j}})$$

but we will seek a far-field ($|\mathbf{r}| \gg |\mathbf{r}'|| = R$) solution only.



The vector potential $\mathbf{A}(\mathbf{x})$ (3.17) expands as

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{\mu_0 I}{4\pi}\oint \frac{dr'}{|\mathbf{r}-\mathbf{r}'|} \\ &= \frac{\mu_0}{4\pi}\oint d\mathbf{r}'\left(\frac{1}{r}+\frac{\mathbf{r}\cdot\mathbf{r}'}{r^3}+...\right)\end{aligned} \tag{3.20}$$

under localized current. Also the integral involving $\frac{1}{r}$ vanishes around the loop. So

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^3}\oint \mathbf{r}\cdot\mathbf{r}' d\mathbf{r}' = \frac{-\mu_0 I}{4\pi r^3}\int_S \nabla_{\mathbf{r}'}(\mathbf{r}\cdot\mathbf{r}')\times dS$$

because of Green's theorem

$$\oint_C f d\mathbf{r} = \int \nabla f \times d\mathbf{S}.$$

The above is then equal to

$$\frac{-\mu_0 I}{4\pi r^3}\int_S \mathbf{r}\times d\mathbf{S} = \frac{\mu_0 I}{4\pi r^3}\mathbf{r}\times\int d\mathbf{S}$$

since $\mathbf{r}$ is juts a constant vector in this integral. Now the integral of $d\mathbf{S}$ is the vector area $\mathbf{S}$ of surface S. So the above is equal to

$$-\frac{\mu_0 I}{4\pi}\frac{\mathbf{r}\times\mathbf{S}}{r^3}$$

Define a *magnetic dipole moment* by

$$\mathbf{m} = I\mathbf{S} \tag{3.21}$$

and far field is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3}. \tag{3.22}$$

Exercise Show that the magnetic dipole field $\mathbf{B}(\mathbf{r})$ takes an identical form to the electric dipole (2.18), i.e.,

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\left(\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3}\right) \tag{3.23}$$

3.4.1 General magnetic field solutions

For a general current distribution $\mathbf{J}(\mathbf{x})$, note the following identities (note summation conventions)

$$\frac{\partial}{\partial x_i}(J_i x_j) = \frac{\partial J_i}{\partial x_i}x_j + J_i\delta_{ij} = J_j \tag{*}$$

since the first term is zero by the continuity equation $\nabla\cdot\mathbf{J} = 0$. So $\mathbf{J}$ can be expressed as a total derivative

$$\frac{\partial}{\partial x_i}(J_i x_j x_k) = \frac{\partial J_i}{\partial x_i}x_j x_k + J_j x_k + J_k x_j = J_j x_k + J_k x_j \tag{†}$$

So the general solution (3.17) becomes

$$A_i(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3\mathbf{x}'\frac{J_i(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \frac{\mu_0}{4\pi}\int d^3\mathbf{x}' \times \left(\frac{J_i(\mathbf{x}'}{r} + \frac{J_i(\mathbf{x}'(\mathbf{x}\cdot\mathbf{x}')}{r^3} + \ldots\right)$$
$$= \frac{\mu_0}{4\pi}\left\{\frac{1}{r}\int d^3\mathbf{x}'\frac{\partial}{\partial x_j}(J_j x_i') + \frac{x_j}{r^3}\int d^3\mathbf{x}'\left[\frac{1}{2}J_i x_j' + \frac{1}{2}J_i x_i' + \frac{1}{2}J_i x_j' - \frac{1}{2}J_i x_i'\right]\right\}$$

for the first term, by (*), the surface term

$$\sim \frac{1}{r}\int_S x_i' J_i dS_j = 0$$

vanishes with $V \subset V$ interior sources ($r' \ll r$). And for the second term, by (†) the surface term is $\frac{1}{2}\frac{\partial}{\partial x_i}(J_k x_i' x_j')$. So above is equal to

$$\frac{\mu_0}{4\pi r^3}\frac{x_j}{2}\int d^3\mathbf{x}'(J_i x_j' - J_j x_i')$$
$$=\frac{\mu_0}{4\pi r^3}\frac{1}{2}\int d^3\mathbf{x}'\left[J_i(\mathbf{x}\cdot\mathbf{x}') - x_i'(\mathbf{J}\cdot\mathbf{x})\right]$$
$$= \frac{-\mu_0}{4\pi r^3}\frac{1}{2}\left[\mathbf{x}\times\int d^3\mathbf{x}' J(\mathbf{x}'\times\mathbf{x}'\right]$$

Hence

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3} \tag{3.24}$$

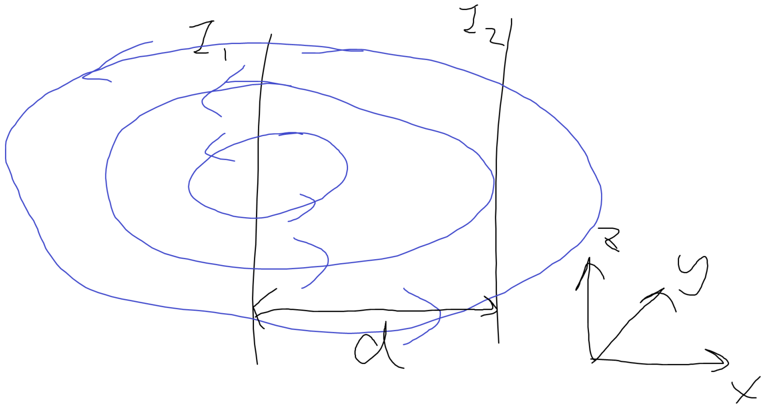
where

$$\mathbf{m} = \frac{1}{2}\int d^3\mathbf{r}'(\mathbf{r}'\times\mathbf{J}(\mathbf{r}')) \tag{3.25}$$

3.5 Magnetic Forces

Ampere showed that one current-carrying wire (current I_1) exerts a force on a second wire (I_2), so consider the force on the second wire in the $\mathbf{B}-$field of the first.

3.5.1 Two straight wires



Parallel to $z-$axis, distance d apart. So we have (3.5),

$$B_1 = \frac{\mu_0 I_1}{2\pi r}\hat{\varphi}.$$

Also, $\mathbf{J}_2 = nq\mathbf{v}$, where n is the density of charge carriers and $\mathbf{v}$ is the average velocity in the $z-$direction, and $I_2 = J_2 A$, where A is the cross-sectional area of wire.

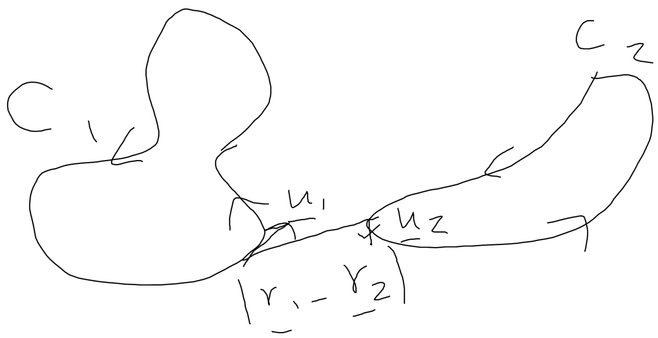
From the Lorentz force $\mathbf{F} = q\mathbf{v}\times\mathbf{B}$, we get a force law per unit length,

$$\mathbf{f} = nA\mathbf{F} = nAq\mathbf{v}\times\mathbf{B}_1 = A\mathbf{J}_2\times\mathbf{B}_1 = \mu_0\frac{I_1 I_2}{2\pi d}\hat{\mathbf{z}}\times\hat{\varphi} = -\mu_0\frac{I_1 I_2}{2\pi d}\hat{\mathbf{x}} \tag{3.26}$$

where nA is the number of charge carriers per unit length.

If I_1 and I_2 have the *same* direction ($I_1 I_2 > 0$), then the force is attractive. Conversely, the force is repulsive.

3.6 General case



First loop on curve C_1 with current I_1, and line element $d\mathbf{r}_1$ induces:

$$\mathbf{B}_1(\mathbf{r}) = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} \frac{d\mathbf{r} + 1 \times (\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3}.$$

Integrated force on second loop

$$\mathbf{F} = \int d^3\mathbf{r} J_2(\mathbf{r}) \times \mathbf{B}_1(\mathbf{r}) \tag{3.27}$$

$$\begin{aligned} &= I_2 \oint d\mathbf{r}_2 \times B\mathbf{B}_1(\mathbf{r}) \\ &= \frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\mathbf{r}_2 \times (d\mathbf{r}_1 \times (\mathbf{r}_2 - \mathbf{r}_1))}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \end{aligned} \tag{3.28}$$

Suppose loops are well-separated ($r = |\mathbf{r}_2 - \mathbf{r}_1| > R_1, R_2$). Expand to find

$$\mathbf{F} = \frac{\mu_0}{4\pi} \nabla \left(\frac{3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}}) - \mathbf{m}_1 \cdot \mathbf{m}_2}{r^3} \right) \tag{3.29}$$

(See D Tong's EM notes – non-examinable).

4 Electrodynamics

4.1 Faraday's Law of Induction

Consider the time-dependent Maxwell's equations (Faraday's law of induction, 1.14),

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.1}$$

This shows how varying magnetic fields *induces* electric fields (and, in turn, currents and further magnetic fields will be created, (Gauss' Law, 1.12), (Ampere-Maxwell Law, 1.15)).

(missing 1 lecture)

Inductance (4.8) $L = \phi/I$.

Solenoid (continued)

Field through single turn $B = \mu_0 IN$ and flux is $\phi_0 = \mu_0 INA$ and total flux is

$$\phi = \phi_0 NL = \mu_0 IN^2 Al = \mu_0 IN^2 \nu$$

So self-inductance is

$$L = \phi/I = \mu_0 N^2 \nu \tag{4.9}$$

Work must be done to create I but this is reversible.

4.1.1 Magnetostatic energy

How much energy is stored in wire curve C with current I? Build up from $U = 0$ and use inductance L to find the work done.

Change in current $\frac{dI}{dt}$ induces EMF because of flux change (4.8)

$$\varepsilon = -\frac{d\phi}{dt} = -L\frac{dI}{dt} \tag{4.10}$$

The current must do work (recall $dV/dt = P = VI$)

$$\delta W = \varepsilon I \delta t = -LI\frac{dI}{dt}\delta t$$

So $dW/dt = -LIdI/dt$ which integrates to

$$W = \frac{1}{2}LU^2 = \frac{1}{2}I\phi \tag{4.11}$$

Example. Consider solenoid with $|phi = \mu_0 IN^2\nu$, we have

$$W = \frac{1}{2}I\phi = \frac{1}{2}\mu_0 I^2 N^2 \nu = \frac{1}{2\mu_0}B^2\nu$$

The energy of steady current is stored in magnetic fields:

$$\begin{aligned} U &= \frac{1}{2}I\phi = \frac{1}{2}I\int_S \mathbf{B}\cdot d\mathbf{S} = \frac{1}{2}I\int_S \nabla\times\mathbf{A}\cdot d\mathbf{S} = \frac{1}{2}\oint_C \mathbf{A}\cdot d\mathbf{r} \\ &= \frac{1}{2}\int_V d^3\mathbf{r}\mathbf{J}\cdot\mathbf{A} \\ &= \frac{1}{2\mu_0}\int d^3\mathbf{r}\nabla\times\mathbf{B}\cdot\mathbf{A} \\ &= \frac{1}{2\mu_0}\int d^3\mathbf{r}[\nabla\cdot(\mathbf{B}\times\mathbf{A}) + \mathbf{B}\cdot\nabla\times\mathbf{A}] \end{aligned}$$

note that we've used Stoke's theorem, Maxwell's theorem (3.1) and the identity

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\cdot\nabla\times\mathbf{B} - \mathbf{B}\cdot\nabla\times\mathbf{A}$$

So the above equals

$$\frac{1}{2\mu_0}\int d^3\mathbf{r}\mathbf{B}\cdot\mathbf{B} \tag{4.12}$$

Also applies for several curves C_i, currents I_i. Combining this with electrostatic energy (2.27) we have

$$U = \int d^3\mathbf{r}\left(\frac{\varepsilon_0}{2}\mathbf{E}\cdot\mathbf{E} + \frac{1}{2\mu_0}\mathbf{B}\cdot\mathbf{B}\right) \tag{4.13}$$

4.2 Resistance and heat loss

Building up (or maintaining) current I also requires irreversible work because of friction or resistance. Usually there is an effective EMF $\mathcal{E}$ proportional to the speed of the charge carriers: Ohm's law is

$$\mathcal{E} = IR \tag{4.14}$$

where R is the resistance of the circuit C.

For a wire of cross-sectional area A, length l the *resistivity* ρ is

$$\rho = \frac{AR}{l} \tag{4.15}$$

while the *conductivity* σ is $\sigma = 1/\rho$. In general, Ohm's Law is

$$\mathbf{J} = \sigma\mathbf{E} \tag{4.16}$$

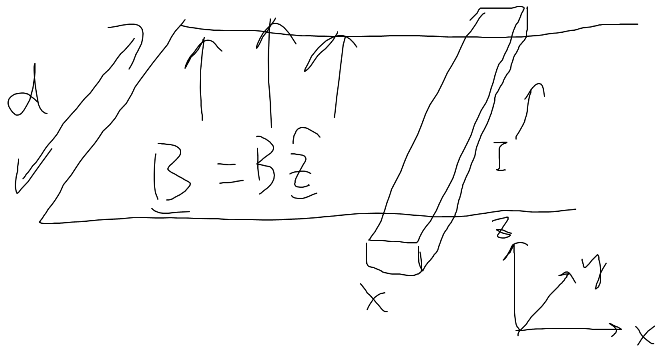
4.2.1 Energy dissipation (Joule heating)

In the presence of resistance, work is required to maintain a current I. In a time δt,

$$\delta W = \mathcal{E}I\delta t = I^2R\delta t \implies \frac{dW}{dt} = I^2R \tag{4.17}$$

This energy is lost as friction or heating.

4.2.2 Moving wire example



Suppose we have a frictionless sliding bar (length d, mass m). Degrees of freedom position x, current I. For position, the Lorentz force per unit length is

$$\mathbf{f} = IB\hat{\mathbf{y}} \times \hat{\mathbf{z}}$$

so total force

$$\mathbf{F} = IBd\hat{\mathbf{x}}$$

From $\mathbf{F} = m\ddot{x}$ we have $m\ddot{x} = IBd$ (*) (we ignore B due to the current itself here).

For current, we know total EMF is

$$\mathcal{E} = -\frac{d\phi}{dt} = -Bdv = -Bd\dot{x}$$

But Ohm's Law (4.14) gives $I = \mathcal{E}/R = -Bd\dot{x}/R$. So we have

$$m\ddot{x} = -B^2d^2\dot{x}/R$$

which has the decaying solution

$$\dot{x} = -v_0 e^{-B^2d^2t/mR} \tag{+}$$

where v_0 is the initial velocity. Whichever way the bar moves by Lenz's law acts against the motion. Current obeys

$$I = \mathcal{E}/R = -Bd\dot{x}$$

so the energy dissipates

$$dW/dt = \mathcal{E}I = I^2R. \tag{4.17}$$

With a battery with EMF $\mathcal{E}$ included a current $I_0 = \varepsilon_0/R$, the total EMF becomes

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_{induced} = \mathcal{E}_0 - Bd\dot{x}$$

Again using Ohm's Law $\mathcal{E} = IR$ we have

$$m\ddot{x} = IBd = -Bd/R(Bd\dot{x} - \mathcal{E}_0)$$

This is simple to solve exploiting the solution (+).